Topological structures in Colombeau algebras: topological $\widetilde{\mathbb{C}}$-modules and duality theory

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Abstract

We study modules over the ring $\widetilde{\mathbb{C}}$ of complex generalized numbers from a topological point of view, introducing the notions of $\widetilde{\mathbb{C}}$-linear topology and locally convex $\widetilde{\mathbb{C}}$-linear topology. In this context particular attention is given to completeness, continuity of $\widetilde{\mathbb{C}}$-linear maps and elements of duality theory for topological $\widetilde{\mathbb{C}}$-modules. As main examples we consider various Colombeau algebras of generalized functions.

Key words: modules over the ring of complex generalized numbers, algebras of generalized functions, topology, duality theory

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0 Introduction

Colombeau algebras of generalized functions have proved to be an analytically powerful tool in dealing with linear and nonlinear PDEs with highly singular coefficients [1, 2, 3, 8, 9, 10, 18, 21, 23, 24, 29, 30, 32, 33, 34, 36, 37, 41]. In the recent research on the subject a variety of algebras of generalized functions [3, 12, 13, 17, 19, 22] have been introduced in addition to the original construction by Colombeau [6, 7] and investigated in its algebraic and structural aspects as well as in analytic and applicative aspects. These investigations have produced a theory of point values in the Colombeau algebra $\mathcal{G}(\Omega)$ and results of invertibility and positivity in the ring of constant generalized functions $\widetilde{\mathbb{C}}$ [19, 38, 39] but also microlocal analysis in Colombeau algebras and regularity theory for generalized solutions to partial and (pseudo-) differential equations [11, 13, 15, 16, 20, 21, 22, 23, 24, 25, 26, 27]. Apart from some early and inspiring work by Biagioni,

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Pilipović, Scarpalézos [1, 32, 46, 47, 48], topological questions have played a marginal role in the existing Colombeau literature. However, the recent papers on pseudodifferential operators acting on algebras of generalized functions [13, 15, 16] and a preliminary kernel theory introduced in [15] motivate a renewed interest in topological issues, such as $\widetilde{\mathbb{C}}$-linear topologies on Colombeau algebras, $\widetilde{\mathbb{C}}$-linear continuous maps and duality theory.

This is the first of two papers devoted to a detailed topological investigation into algebras of generalized functions and succeeds to add a collection of original results to what is already known in the field. It develops a theory of topological $\widetilde{\mathbb{C}}$-modules and locally convex topological $\widetilde{\mathbb{C}}$-modules, which requires the introduction of $\widetilde{\mathbb{C}}$-versions of various concepts relating to topological and locally convex vector spaces. As a topic of particular interest, the foundations of duality theory are provided within this framework, dealing with the $\widetilde{\mathbb{C}}$-module $\mathcal{L}(\mathcal{G},\widetilde{\mathbb{C}})$ of all $\widetilde{\mathbb{C}}$-linear and continuous functionals on $\mathcal{G}$. The second paper on topological structures in Colombeau algebras [14] will be focused on applications. Due to the fact that many algebras of generalized functions can be easily viewed as locally convex topological $\widetilde{\mathbb{C}}$-modules, we will be able to apply all the previous theoretical concepts and results to the *topological dual of a Colombeau algebra*. This procedure together with the discussion of some relevant examples and continuous embeddings is a novelty in Colombeau theory.

We now describe the contents of the sections in more detail.

Section 1 serves to collect the basic topological notions which we will refer to in the course of the paper. Starting from the new notions of $\widetilde{\mathbb{C}}$- absorbent, balanced and convex subsets of a $\widetilde{\mathbb{C}}$-module $\mathcal{G}$, $\widetilde{\mathbb{C}}$-linear and locally convex $\widetilde{\mathbb{C}}$-linear topologies are introduced and described via their neighborhoods in Subsections 1.1 and 1.2 respectively. A characterization of locally convex topological $\widetilde{\mathbb{C}}$-modules is given, inspired by the analogous statements involving seminorms and locally convex vector spaces, making use of the concept of ultra-pseudo-seminorm. This turns out to be a useful technical tool in providing the classically expected results on separatedness and boundedness. In particular, the continuity of a $\widetilde{\mathbb{C}}$-linear map is expressed in terms of a uniform estimate between ultra-pseudo-seminorms. Inductive limits and strict inductive limits of locally convex topological $\widetilde{\mathbb{C}}$-modules are studied in Subsection 1.3. Finally Subsection 1.4 is concerned with completeness in topological $\widetilde{\mathbb{C}}$-modules. We pay particular attention to the relationships between completeness, strict inductive limit topology and initial topology in case of locally convex topological $\widetilde{\mathbb{C}}$-modules.

The theoretical core of the paper is Section 2 where we set the stage for the duality theory of topological $\widetilde{\mathbb{C}}$-modules. Using concepts as pairings of $\widetilde{\mathbb{C}}$-modules and polar sets we equip the dual $\mathcal{L}(\mathcal{G},\widetilde{\mathbb{C}})$ with at least three locally convex $\widetilde{\mathbb{C}}$-linear topologies: the weak topology $\sigma(\mathcal{L}(\mathcal{G},\widetilde{\mathbb{C}}),\mathcal{G})$, the strong topology $\beta(\mathcal{L}(\mathcal{G},\widetilde{\mathbb{C}}),\mathcal{G})$ and the topology $\beta_b(\mathcal{L}(\mathcal{G},\widetilde{\mathbb{C}}),\mathcal{G})$ of uniform convergence on bounded subsets of $\mathcal{G}$. A theorem of completeness of the dual $\mathcal{L}(\mathcal{G},\widetilde{\mathbb{C}})$ with respect to the strong topology as well as a $\widetilde{\mathbb{C}}$-linear formulation of the Banach-Steinhaus theorem are obtained under suitable hypotheses on $\mathcal{G}$.

Section 3 investigates the properties of some interesting examples of locally convex topological $\widetilde{\mathbb{C}}$-modules and their topological duals. Inspired by [47] Sub-

section 3.1 deals with the $\widetilde{\mathbb{C}}$-modules $\mathcal{G}_E$ of generalized functions based on the locally convex topological vector space E, showing how a separated locally convex $\widetilde{\mathbb{C}}$-linear topology may be defined, in terms of ultra-pseudo-seminorms, on $\mathcal{G}_E$ by means of the seminorms which topologize E. Well-known Colombeau algebras as $\widetilde{\mathbb{C}}$, $\mathcal{G}(\Omega)$ [19] and $\mathcal{G}_{\mathscr{S}}(\mathbb{R}^n)$ [13, 15] are recognized to be special cases. More sophisticated topological tools, as strict inductive limit topologies and initial topologies, are needed for the Colombeau algebra of compactly supported generalized functions $\mathcal{G}_{\mathrm{c}}(\Omega)$ [19] and the Colombeau algebra of tempered generalized functions $\mathcal{G}_\tau(\mathbb{R}^n)$ [19]. Finally, ultra-pseudo seminorms and norms fitted to measure the regularity of generalized functions are introduced, providing a topology for the Colombeau algebras $\mathcal{G}^\infty(\Omega)$, $\mathcal{G}^\infty_{\mathrm{c}}(\Omega)$, $\mathcal{G}^\infty_{\mathscr{S}}(\mathbb{R}^n)$ [13, 15]. The continuity of $\widetilde{\mathbb{C}}$-linear maps of the form $T : \mathcal{G}_E \to \mathcal{G}$ is the topic of Subsection 3.2, while Subsection 3.3 is devoted to the topological dual $\mathrm{L}(\mathcal{G}_E, \widetilde{\mathbb{C}})$ when E is a normed space. In this particular case, an ultra-pseudo-norm modelled on the classical dual norm $\|\cdot\|_{E'}$ is defined on $\mathrm{L}(\mathcal{G}_E, \widetilde{\mathbb{C}})$ and a generalization of the Hahn-Banach theorem is given. This result combined with a further adaptation of the Banach-Steinhaus theorem to the context of ultra-pseudo-normed $\widetilde{\mathbb{C}}$-modules allows to compare different $\widetilde{\mathbb{C}}$-linear topologies on $\mathrm{L}(\mathcal{G}_E, \widetilde{\mathbb{C}})$.

1 Topological $\widetilde{\mathbb{C}}$-modules

This section provides the required foundations of topology for $\widetilde{\mathbb{C}}$-modules. We begin with a collection of basic notions and definitions.

Let $\widetilde{\mathbb{C}}$ be the *ring of complex generalized numbers* obtained factorizing

$$\mathcal{E}_M := \{(u_\varepsilon)_\varepsilon \in \mathbb{C}^{(0,1]} : \exists N \in \mathbb{N} \quad |u_\varepsilon| = O(\varepsilon^{-N}) \text{ as } \varepsilon \to 0\}$$

with respect to the ideal

$$\mathcal{N} := \{(u_\varepsilon)_\varepsilon \in \mathbb{C}^{(0,1]} : \forall q \in \mathbb{N} \quad |u_\varepsilon| = O(\varepsilon^{q}) \text{ as } \varepsilon \to 0\}$$

(c.f. [7, 19]). $\widetilde{\mathbb{C}}$ is trivially a module over itself and it can be endowed with a structure of a topological ring. In order to explain this assertion, inspired by nonstandard analysis [40, 50] and the previous work in this field [1, 32, 46, 47, 48], we introduce the function

$$\mathrm{v} : \mathcal{E}_M \to (-\infty, +\infty] : (u_\varepsilon)_\varepsilon \to \sup\{b \in \mathbb{R} : \quad |u_\varepsilon| = O(\varepsilon^b) \text{ as } \varepsilon \to 0\} \tag{1.1}$$

on $\mathcal{E}_M$. It satisfies the following conditions:

(i) $\mathrm{v}((u_\varepsilon)_\varepsilon) = +\infty$ if and only if $(u_\varepsilon)_\varepsilon \in \mathcal{N}$,

(ii) $\mathrm{v}((u_\varepsilon)_\varepsilon (v_\varepsilon)_\varepsilon) \geq \mathrm{v}((u_\varepsilon)_\varepsilon) + \mathrm{v}((v_\varepsilon)_\varepsilon)$,

(iii) $\mathrm{v}((u_\varepsilon)_\varepsilon + (v_\varepsilon)_\varepsilon) \geq \min\{\mathrm{v}((u_\varepsilon)_\varepsilon), \mathrm{v}((v_\varepsilon)_\varepsilon)\}$,

where *(ii)* and *(iii)* become equality if at least one or both terms are of the form $(c\varepsilon^b)_\varepsilon$, $c \in \mathbb{C}$, $b \in \mathbb{R}$, respectively. Note that if $(u_\varepsilon - u'_\varepsilon)_\varepsilon \in \mathcal{N}$, *(i)* combined

with (iii) yields $\mathrm{v}((u_\varepsilon)_\varepsilon) = \mathrm{v}((u'_\varepsilon)_\varepsilon)$. This means that we can use (1.1) to define the *valuation*

$$\mathrm{v}_{\widetilde{\mathbb{C}}}(u) := \mathrm{v}((u_\varepsilon)_\varepsilon) \tag{1.2}$$

of the complex generalized number $u = [(u_\varepsilon)_\varepsilon]$, and that all the previous properties hold for the elements of $\widetilde{\mathbb{C}}$. Let now

$$|\cdot|_{\mathrm{e}} := \widetilde{\mathbb{C}} \to [0, +\infty) : u \to |u|_{\mathrm{e}} := \mathrm{e}^{-\mathrm{v}_{\widetilde{\mathbb{C}}}(u)}. \tag{1.3}$$

The properties of the valuation on $\widetilde{\mathbb{C}}$ makes the coarsest topology on $\widetilde{\mathbb{C}}$ such that the map $|\cdot|_{\mathrm{e}}$ is continuous compatible with the ring structure. It is common in the already existing literature [32, 46, 47, 48] to use the adjective "sharp" for such a topology. In this paper $\widetilde{\mathbb{C}}$ will always be endowed with its "sharp topology". Our investigation of the topological aspects of a $\widetilde{\mathbb{C}}$-module is mainly modeled on the classical approach to topological vector spaces and locally convex spaces suggested by many books on functional analysis [28, 42]. In particular it requires the adaptation of the algebraic notions of absorbent, balanced and convex subsets of a vector space, to the new context of $\widetilde{\mathbb{C}}$-modules.

Definition 1.1. *A subset A of a $\widetilde{\mathbb{C}}$-module $\mathcal{G}$ is $\widetilde{\mathbb{C}}$-absorbent if for all $u \in \mathcal{G}$ there exists $a \in \mathbb{R}$ such that $u \in [(\varepsilon^b)_\varepsilon]A$ for all $b \le a$.*
$A \subseteq \mathcal{G}$ is $\widetilde{\mathbb{C}}$-balanced if $\lambda A \subseteq A$ for all $\lambda \in \widetilde{\mathbb{C}}$ with $|\lambda|_{\mathrm{e}} \le 1$.
$A \subseteq \mathcal{G}$ is $\widetilde{\mathbb{C}}$-convex if $A + A \subseteq A$ and $[(\varepsilon^b)_\varepsilon]A \subseteq A$ for all $b \ge 0$.

Note that if A contains 0 then it is $\widetilde{\mathbb{C}}$-convex if and only if $[(\varepsilon^{b_1})_\varepsilon]A + [(\varepsilon^{b_2})_\varepsilon]A \subseteq A$ for all $b_1, b_2 \ge 0$. A subset A which is both $\widetilde{\mathbb{C}}$-balanced and $\widetilde{\mathbb{C}}$-convex is called absolutely $\widetilde{\mathbb{C}}$-convex. In the case when A is $\widetilde{\mathbb{C}}$-balanced the convexity is equivalent to the following statement: for all $\lambda, \mu \in \widetilde{\mathbb{C}}$ with $\max\{|\lambda|_{\mathrm{e}}, |\mu|_{\mathrm{e}}\} \le 1$, $\lambda A + \mu A \subseteq A$. The $\widetilde{\mathbb{C}}$-convexity cannot be considered as a generalization of the corresponding concept in vector spaces. In fact the only subset A of $\mathbb{C}$ which is $\widetilde{\mathbb{C}}$-convex is the trivial set $\{0\}$.

Remark 1.2. The definition of a $\widetilde{\mathbb{C}}$-balanced subset of $\mathcal{G}$ is inspired by the classical one concerning vector spaces and consists in replacing the absolute value in $\mathbb{C}$ with $|\cdot|_{\mathrm{e}}$ in $\widetilde{\mathbb{C}}$. We may construct an analogy between a vector space V and a $\widetilde{\mathbb{C}}$-module $\mathcal{G}$ by associating the sum in V with the sum in $\mathcal{G}$ and the product au, $a > 0$, $u \in V$, with $[(\varepsilon^{-\log a})_\varepsilon]u$ where $u \in \mathcal{G}$. In this way the concept of absorbent subset of V is translated into the concept of $\widetilde{\mathbb{C}}$-absorbent subset of $\mathcal{G}$ and a convex cone (at 0) in V corresponds to a $\widetilde{\mathbb{C}}$-convex subset of $\mathcal{G}$.

In the sequel, we shall simply talk about absorbent, balanced or convex subset, omitting the prefix $\widetilde{\mathbb{C}}$, when we deal with $\widetilde{\mathbb{C}}$-modules. The reader should be aware that the words refer to Definition 1.1 and not to the classical notions in this context.

1.1 Elementary properties of $\widetilde{\mathbb{C}}$-linear topologies

We recall that a topology τ on a $\widetilde{\mathbb{C}}$-module $\mathcal{G}$ is said to be $\widetilde{\mathbb{C}}$-linear if the addition $\mathcal{G} \times \mathcal{G} \to \mathcal{G} : (u, v) \to u + v$ and the product $\widetilde{\mathbb{C}} \times \mathcal{G} \to \mathcal{G} : (\lambda, u) \to$

λu are continuous. A topological $\widetilde{\mathbb{C}}$-module $\mathcal{G}$ is a $\widetilde{\mathbb{C}}$-module with a $\widetilde{\mathbb{C}}$-linear topology. As an immediate consequence we have that for any $u_0 \in \mathcal{G}$ and for any invertible $\lambda \in \widetilde{\mathbb{C}}$ the translation $\mathcal{G} \to \mathcal{G} : u \to u + u_0$ and the mapping $\mathcal{G} \to \mathcal{G} : u \to \lambda u$ are homeomorphisms of $\mathcal{G}$ into itself. This means that if $\mathcal{U}$ is a base of neighborhoods of the origin that $\mathcal{U} + u_0$ is a base of neighborhoods of u_0 and if U is a neighborhood of the origin so is λU for all invertible $\lambda \in \widetilde{\mathbb{C}}$. It is also clear that a $\widetilde{\mathbb{C}}$-linear map T between topological $\widetilde{\mathbb{C}}$-modules is continuous if and only if it is continuous at the origin and then the set $\mathrm{L}(\mathcal{G}, \mathcal{H})$ of all continuous $\widetilde{\mathbb{C}}$-linear maps between the topological $\widetilde{\mathbb{C}}$-modules $\mathcal{G}$ and $\mathcal{H}$ is a module on $\widetilde{\mathbb{C}}$.

Proposition 1.3. *Let $\mathcal{G}$ be a topological $\widetilde{\mathbb{C}}$-module and $\mathcal{U}$ be a base of neighborhoods of the origin. Then for each $U \in \mathcal{U}$,*

(i) U is absorbent,

(ii) there exists $V \in \mathcal{U}$ with $V + V \subseteq U$,

(iii) there exists a balanced neighborhood of the origin W such that $W \subseteq U$.

Proof. Fix $u \in \mathcal{G}$. The continuity of the product between elements of $\widetilde{\mathbb{C}}$ and elements of $\mathcal{G}$ guarantees for any $U \in \mathcal{U}$ the existence of $\eta > 0$ such that $\lambda u \in U$ for all $\lambda \in \widetilde{\mathbb{C}}$ with $|\lambda|_{\mathrm{e}} \leq \eta$. Hence $u \in [(\varepsilon^b)_\varepsilon]U$ provided $b \leq \log \eta$. This shows that U is absorbent.
The addition in $\mathcal{G}$ is continuous. Therefore, given $U \in \mathcal{U}$ there exist $V_1, V_2 \in \mathcal{U}$ such that $V_1 + V_2 \subseteq U$ and a neighborhood $V \in \mathcal{U}$ contained in $V_1 \cap V_2$ which proves assertion (ii).
Finally, since the product is continuous at $(0,0)$, there exist $\eta > 0$ and $V \in \mathcal{U}$ such that $\lambda V \subseteq U$ for $|\lambda|_{\mathrm{e}} \leq \eta$. Let $W = \cup_{|\lambda|_{\mathrm{e}} \leq \eta} \lambda V$. By construction W is contained in U and is a neighborhood of the origin since $[(\varepsilon^{-\log \eta})_\varepsilon]V \subseteq W$. Recalling that $|\lambda\mu|_{\mathrm{e}} \leq |\lambda|_{\mathrm{e}}|\mu|_{\mathrm{e}}$ for all complex generalized numbers λ, μ, we conclude that W is a balanced subset of $\mathcal{G}$. □

It follows from Proposition 1.3 that any topological $\widetilde{\mathbb{C}}$-module has a base of absorbent and balanced neighborhoods of the origin. As in the classical theory of topological vector spaces this fact ensures a useful characterization of separated topological $\widetilde{\mathbb{C}}$-modules. Further, if $\mathcal{G}$ is a topological $\widetilde{\mathbb{C}}$-module and $\mathcal{U}$ a base of neighborhoods of the origin we have that $\mathcal{G}$ is separated if and only if $\cap_{U \in \mathcal{U}} U = \{0\}$.

Remark 1.4. A particular example of a topological $\widetilde{\mathbb{C}}$-module is the quotient set $\mathcal{G}/M$, where M is a $\widetilde{\mathbb{C}}$-submodule of the topological $\widetilde{\mathbb{C}}$-module $\mathcal{G}$, endowed with the quotient topology. In analogy with the theory of topological vector spaces, by the definition of a quotient $\widetilde{\mathbb{C}}$-module $\mathcal{G}/M$ and the previous considerations on separated modules we obtain that $\mathcal{G}/M$ is separated if and only if M is closed in $\mathcal{G}$.

Finally, having introduced a notion of absorbent set we can state the natural definition of a bounded subset of a topological $\widetilde{\mathbb{C}}$-module.

Definition 1.5. *We say that a subset A of a topological $\widetilde{\mathbb{C}}$-module $\mathcal{G}$ is bounded if it is absorbed by every neighborhood of the origin i.e. for all neighborhoods U of the origin in $\mathcal{G}$ there exists $a \in \mathbb{R}$ such that $A \subseteq [(\varepsilon^b)_\varepsilon]U$ for all $b \leq a$.*

A simple application of the definitions shows that any continuous $\widetilde{\mathbb{C}}$-linear map $T : \mathcal{G} \to \mathcal{H}$ between topological $\widetilde{\mathbb{C}}$-modules is *bounded*, in the sense that it maps bounded subsets of $\mathcal{G}$ into bounded subsets of $\mathcal{H}$.

1.2 Locally convex topological $\widetilde{\mathbb{C}}$-modules: ultra-pseudo-seminorms and continuity

Definition 1.6. *A locally convex topological $\widetilde{\mathbb{C}}$-module is a topological $\widetilde{\mathbb{C}}$-module which has a base of $\widetilde{\mathbb{C}}$-convex neighborhoods of the origin.*

Proposition 1.3 shows that there exist bases of convex neighborhoods of the origin with additional properties.

Proposition 1.7. *Every locally convex topological $\widetilde{\mathbb{C}}$-module $\mathcal{G}$ has a base of absolutely convex and absorbent neighborhoods of the origin.*

Proof. Let $\mathcal{U}$ be a base of convex neighborhoods of 0 in $\mathcal{G}$. By Proposition 1.3, for all $U \in \mathcal{U}$ there exists a a balanced neighborhood of the origin W contained in U. Take the convex hull W' of W i.e. the set of all finite $\widetilde{\mathbb{C}}$-linear combinations of the form $[(\varepsilon^{b_1})_\varepsilon]w_1 + [(\varepsilon^{b_2})_\varepsilon]w_2 + ... + [(\varepsilon^{b_n})_\varepsilon]w_n$ where $b_i \geq 0$ and $w_i \in W$. By construction W' is an absolutely convex and absorbent neighborhood of 0 and since U is itself convex we have that $W' \subseteq U$. $\square$

We now want to deduce some more information on the topology of $\mathcal{G}$ from the nature of the neighborhoods. We begin with some preliminary definitions and results.

Definition 1.8. *Let $\mathcal{G}$ be a $\widetilde{\mathbb{C}}$-module. A* valuation *on $\mathcal{G}$ is a function $\mathrm{v} : \mathcal{G} \to (-\infty, +\infty]$ such that*

(i) $\mathrm{v}(0) = +\infty$,

(ii) $\mathrm{v}(\lambda u) \geq \mathrm{v}_{\widetilde{\mathbb{C}}}(\lambda) + \mathrm{v}(u)$ *for all* $\lambda \in \widetilde{\mathbb{C}}$, $u \in \mathcal{G}$,

(ii)' $\mathrm{v}(\lambda u) = \mathrm{v}_{\widetilde{\mathbb{C}}}(\lambda) + \mathrm{v}(u)$ *for all* $\lambda = [(c\varepsilon^a)_\varepsilon]$, $c \in \mathbb{C}$, $a \in \mathbb{R}$, $u \in \mathcal{G}$,

(iii) $\mathrm{v}(u + v) \geq \min\{\mathrm{v}(u), \mathrm{v}(v)\}$.

An ultra-pseudo-seminorm *on $\mathcal{G}$ is a function $\mathcal{P} : \mathcal{G} \to [0, +\infty)$ such that*

(i) $\mathcal{P}(0) = 0$,

(ii) $\mathcal{P}(\lambda u) \leq |\lambda|_{\mathrm{e}}\mathcal{P}(u)$ *for all* $\lambda \in \widetilde{\mathbb{C}}$, $u \in \mathcal{G}$,

(ii)' $\mathcal{P}(\lambda u) = |\lambda|_{\mathrm{e}}\mathcal{P}(u)$ *for all* $\lambda = [(c\varepsilon^a)_\varepsilon]$, $c \in \mathbb{C}$, $a \in \mathbb{R}$, $u \in \mathcal{G}$,

(iii) $\mathcal{P}(u + v) \leq \max\{\mathcal{P}(u), \mathcal{P}(v)\}$.

The term valuation has here a slightly different meaning compared to the well-known concept introduced in nonstandard analysis and is deeply connected with the properties of $\mathcal{G}$ as a $\widetilde{\mathbb{C}}$-module. The reader should refer to [31, 40, 43, 44, 50] for the original nonstardard approach and some related applications.

$\mathcal{P}(u) = \mathrm{e}^{-\mathrm{v}(u)}$ is a typical example of an ultra-pseudo-seminorm obtained by means of a valuation on $\mathcal{G}$. An *ultra-pseudo-norm* is an ultra-pseudo-seminorm $\mathcal{P}$ such that $\mathcal{P}(u) = 0$ implies $u = 0$. $|\cdot|_{\mathrm{e}}$ introduced in (1.3) is an ultra-pseudo-norm on $\widetilde{\mathbb{C}}$. We now present an interesting example of a valuation on a $\widetilde{\mathbb{C}}$-module $\mathcal{G}$.

Proposition 1.9. *Let A be an absolutely convex and absorbent subset of a $\widetilde{\mathbb{C}}$-module $\mathcal{G}$. Then*

$$\mathrm{v}_A(u) := \sup\{b \in \mathbb{R} :\ u \in [(\varepsilon^b)_\varepsilon]A\} \tag{1.4}$$

is a valuation on $\mathcal{G}$. Moreover, for $\mathcal{P}_A(u) := \mathrm{e}^{-\mathrm{v}_A(u)}$ and $\eta > 0$ the chain of inclusions

$$\{u \in \mathcal{G} :\ \mathcal{P}_A(u) < \eta\} \subseteq [(\varepsilon^{-\log(\eta)})_\varepsilon]A \subseteq \{u \in \mathcal{G} :\ \mathcal{P}_A(u) \le \eta\} \tag{1.5}$$

holds.

We usually call $\mathcal{P}_A$ the *gauge* of A.

Proof. For each $u \in \mathcal{G}$ the set of real numbers b such that $u \in [(\varepsilon^b)]A$ is not empty. Hence $\mathrm{v}_A(u)$ is clearly a function from $\mathcal{G}$ into $(-\infty, +\infty]$. Since A is balanced, 0 belongs to A and to every $[(\varepsilon^b)_\varepsilon]A$. Thus $\mathrm{v}_A(0) = +\infty$. Assume that $u \in [(\varepsilon^b)]A$ for some $b \in \mathbb{R}$ and write

$$\lambda u = [(\varepsilon^{b+\mathrm{v}_{\widetilde{\mathbb{C}}}(\lambda)})_\varepsilon]\,\lambda\,[(\varepsilon^{-\mathrm{v}_{\widetilde{\mathbb{C}}}(\lambda)})_\varepsilon]\,[(\varepsilon^{-b})_\varepsilon]u,$$

where $\lambda \in \widetilde{\mathbb{C}} \setminus 0$. From $|\lambda\,[(\varepsilon^{-\mathrm{v}_{\widetilde{\mathbb{C}}}(\lambda)})_\varepsilon]|_{\mathrm{e}} = |\lambda|_{\mathrm{e}}\,\mathrm{e}^{\mathrm{v}_{\widetilde{\mathbb{C}}}(\lambda)} = 1$ and the fact that A is a balanced subset of $\mathcal{G}$, we obtain that $\mathrm{v}_A(\lambda u) \ge \mathrm{v}_{\widetilde{\mathbb{C}}}(\lambda) + \mathrm{v}_A(u)$. In particular if λ is of the form $[(c\varepsilon^a)_\varepsilon]$, $c \in \mathbb{C} \setminus 0$, $a \in \mathbb{R}$ and $\lambda u \in [(\varepsilon^b)_\varepsilon]A$, then

$$u = [(\frac{1}{c}\varepsilon^{-a})_\varepsilon]\,[(\varepsilon^b)_\varepsilon]\,[(\varepsilon^{-b})_\varepsilon]\,[(c\varepsilon^a)_\varepsilon]u = [(\varepsilon^{-a+b})_\varepsilon]u'. \tag{1.6}$$

Since $u' = [(\varepsilon^{-b})_\varepsilon][(\varepsilon^a)_\varepsilon]u \in A$, (1.6) leads to $\mathrm{v}_A(u) \ge -\mathrm{v}_{\widetilde{\mathbb{C}}}(\lambda) + \mathrm{v}_A(\lambda u)$ and shows $(ii)'$ in the definition a valuation.
Consider $u, v \in \mathcal{G}$. We know that there exist $b_1, b_2 \in \mathbb{R}$ such that $u \in [(\varepsilon^{b_1})_\varepsilon]A$ and $v \in [(\varepsilon^{b_2})_\varepsilon]A$. The sum $u + v$ is an element of $[(\varepsilon^{b_1}]A + [(\varepsilon^{b_2})_\varepsilon]A$ and we have that $u+v \in [(\varepsilon^{b_1})_\varepsilon](A+[(\varepsilon^{b_2-b_1})_\varepsilon]A)$. Let us assume $b_2 - b_1 \ge 0$ and recall that A is convex. Hence $u + v \in [(\varepsilon^{b_1})_\varepsilon]A$ and $\mathrm{v}_A(u + v) \ge \min\{\mathrm{v}_A(u), \mathrm{v}_A(v)\}$.

Finally, in order to prove (1.5) it is sufficient to observe that $\mathcal{P}_A(u) < \eta$ implies $\mathrm{v}_A(u) > -\log(\eta)$ and $u \in [(\varepsilon^{-\log(\eta)})_\varepsilon]A$ while $u \in [(\varepsilon^{-\log(\eta)})_\varepsilon]A$ implies $\mathrm{v}_A(u) \ge -\log(\eta)$. ∎

Theorem 1.10.

(i) Let $\{\mathcal{P}_i\}_{i\in I}$ be a family of ultra-pseudo-seminorms on a $\widetilde{\mathbb{C}}$-module $\mathcal{G}$. The topology induced by $\{\mathcal{P}_i\}$ on $\mathcal{G}$, i.e. the coarsest topology such that each ultra-pseudo-seminorm is continuous, induces the structure of locally convex topological $\widetilde{\mathbb{C}}$-module on $\mathcal{G}$.

(ii) In a locally convex topological $\widetilde{\mathbb{C}}$-module $\mathcal{G}$ the original topology is induced by the family of ultra-pseudo-seminorms $\{\mathcal{P}_U\}_{U\in\mathcal{U}}$, where $\mathcal{U}$ is a base of absolutely convex and absorbent neighborhoods of the origin.

Proof.

(i) From the properties (ii) and (iii) which characterize an ultra-pseudo-seminorm it is clear that the coarsest topology such that the ultra-pseudo-seminorms $\{\mathcal{P}_i\}_{i\in I}$ are continuous is $\widetilde{\mathbb{C}}$-linear on $\mathcal{G}$. A base of neighborhoods of the origin is given by all the finite intersections of sets of the form $\{u \in \mathcal{G} : \ \mathcal{P}_i(u) \le \eta_i\}$ for $\eta_i > 0$. Each $\{u \in \mathcal{G} : \ \mathcal{P}_i(u) \le \eta_i\}$ is convex. In fact if $\mathcal{P}_i(u_1) \le \eta_i$ and $\mathcal{P}_i(u_2) \le \eta_i$ for all $b_1, b_2 \ge 0$ we have that

$$\begin{aligned}\mathcal{P}_i([(\varepsilon_1^b)_\varepsilon]u_1 + [(\varepsilon^{b_2})_\varepsilon]u_2) &\le \max\{\mathcal{P}_i([(\varepsilon^{b_1})_\varepsilon]u_1), \mathcal{P}_i([(\varepsilon^{b_2})_\varepsilon]u_2)\} \\ &= \max\{\mathrm{e}^{-b_1}\mathcal{P}_i(u_1), \mathrm{e}^{-b_2}\mathcal{P}_i(u_2)\} \le \eta_i.\end{aligned}$$

Since a finite intersection of convex sets is still convex, $\mathcal{G}$ is a locally convex topological $\widetilde{\mathbb{C}}$-module.

(ii) Combining Proposition 1.9 with the previous considerations the topology induced by the family of ultra-pseudo-seminorms $\{\mathcal{P}_U\}_{U\in\mathcal{U}}$ is a locally convex $\widetilde{\mathbb{C}}$-linear topology on $\mathcal{G}$. (1.5) relates the neighborhoods of the origin in this topology with the corresponding neighborhoods in the original topology on $\mathcal{G}$ and shows that the two topologies coincide.

$\square$

Theorem 1.10 and the considerations after Proposition 1.3 lead to the following characterization of separated locally convex topological $\widetilde{\mathbb{C}}$-modules.

Proposition 1.11. *Let $\mathcal{G}$ be a locally convex topological $\widetilde{\mathbb{C}}$-module and $\{\mathcal{P}_i\}_{i\in I}$ a family of continuous ultra-pseudo-seminorms which induces the topology of $\mathcal{G}$. $\mathcal{G}$ is separated if and only if for all $u \ne 0$ there exists $i \in I$ with $\mathcal{P}_i(u) > 0$.*

Example 1.12. If $\mathcal{G}$ is a locally convex topological $\widetilde{\mathbb{C}}$-module and M is a $\widetilde{\mathbb{C}}$-submodule of $\mathcal{G}$ then $\mathcal{G}/M$ equipped with the quotient topology is locally convex itself. Note that if $\mathcal{Q}$ is an ultra-pseudo-seminorm on $\mathcal{G}$ then, denoting the canonical projection of $\mathcal{G}$ on $\mathcal{G}/M$ by π,

$$\dot{\mathcal{Q}}([u]) := \inf_{v\in\pi^{-1}([u])} \mathcal{Q}(v)$$

is a well-defined ultra-pseudo-seminorm on $\mathcal{G}/M$. Indeed, $\dot{\mathcal{Q}}([0]) = 0$ and observing that for all invertible λ in $\widetilde{\mathbb{C}}$, $v \in \pi^{-1}([\lambda u])$ if and only if $\lambda^{-1}v \in \pi^{-1}([u])$ we obtain the estimate

$$\dot{\mathcal{Q}}(\lambda[u]) = \inf_{v\in\pi^{-1}([\lambda u])} \mathcal{Q}(v) \le |\lambda|_{\mathrm{e}} \inf_{v\in\pi^{-1}([u])} \mathcal{Q}(v) = |\lambda|_{\mathrm{e}}\dot{\mathcal{Q}}([u])$$

which becomes an equality when λ is of the form $[(c\varepsilon^b)_\varepsilon] \in \widetilde{\mathbb{C}}$. Finally, consider the sum $[u_1] + [u_2]$. If $\dot{\mathcal{Q}}([u_1]) < \dot{\mathcal{Q}}([u_2])$ then for all $v_2 \in \pi^{-1}([u_2])$ there exists $v_1 \in \pi^{-1}([u_1])$ such that $\mathcal{Q}(v_1) < \mathcal{Q}(v_2)$ and this fact yields

$$\dot{\mathcal{Q}}([u_1] + [u_2]) \le \inf_{\substack{v_1 \in \pi^{-1}([u_1]) \\ v_2 \in \pi^{-1}([u_2])}} \max\{\mathcal{Q}(v_1), \mathcal{Q}(v_2)\} \le \inf_{v_2 \in \pi^{-1}([u_2])} \mathcal{Q}(v_2) = \dot{\mathcal{Q}}([u_2]).$$

If $\dot{\mathcal{Q}}([u_1]) = \dot{\mathcal{Q}}([u_2])$ then for all $v_2 \in \pi^{-1}([u_2])$ and for all $\delta > 0$ there exists $v_1 \in \pi^{-1}([u_1])$ such that $\mathcal{Q}(v_1) \le \mathcal{Q}(v_2) + \delta$. It follows that

$$\dot{\mathcal{Q}}([u_1] + [u_2]) \le \inf_{\substack{v_1 \in \pi^{-1}([u_1]) \\ v_2 \in \pi^{-1}([u_2])}} \max\{\mathcal{Q}(v_1), \mathcal{Q}(v_2)\} \le \inf_{v_2 \in \pi^{-1}([u_2])} \mathcal{Q}(v_2) + \delta$$

and therefore $\dot{\mathcal{Q}}([u_1] + [u_2]) \le \dot{\mathcal{Q}}([u_2])$.

The quotient topology τ on $\mathcal{G}/M$ is determined by the ultra-pseudo-seminorms $\{\dot{\mathcal{Q}}\}_{\mathcal{Q}}$, where $\mathcal{Q}$ is an ultra-pseudo-seminorm continuous on $\mathcal{G}$.

The ultra-pseudo-seminorms provide a useful tool for checking if a subset of a locally convex topological $\widetilde{\mathbb{C}}$-module is bounded. In the sequel let $(\mathcal{G}, \{\mathcal{P}_i\}_{i\in I})$ be a locally convex topological $\widetilde{\mathbb{C}}$-module whose topology is determined by the family of ultra-pseudo-seminorms $\{\mathcal{P}_i\}_{i\in I}$.

Proposition 1.13. *Let $(\mathcal{G}, \{\mathcal{P}_i\}_{i\in I})$ be a locally convex topological $\widetilde{\mathbb{C}}$-module. $A \subseteq \mathcal{G}$ is bounded if and only if for all $i \in I$ there exists a constant $C_i > 0$ such that $\mathcal{P}_i(u) \le C_i$ for all $u \in A$.*

Proof. If $A \subseteq \mathcal{G}$ is bounded then for some $a_i \in \mathbb{R}$ it is contained in the set $[(\varepsilon^{a_i})_\varepsilon]\{u \in \mathcal{G} :\ \mathcal{P}_i(u) \le 1\}$. This means that $\mathcal{P}_i([(\varepsilon^{-a_i})_\varepsilon]u) \le 1$ for all $u \in A$ and the property $(ii)'$ which characterizes an ultra-pseudo-seminorm yields $|[(\varepsilon^{-a_i})_\varepsilon]|_{\mathrm{e}}\mathcal{P}_i(u) \le 1$. Thus $\mathcal{P}_i(u) \le \mathrm{e}^{-a_i}$ for every u in A. Conversely, take a typical neighborhood of the origin of the form $U = \cap_{i\in I_0}\{u \in \mathcal{G} :\ \mathcal{P}_i(u) \le \eta_i\}$ where I_0 is a finite subset of I. Again by the definition of an ultra-pseudo-seminorm and by $\mathcal{P}_i(u) \le C_i$ on A we have that $[(\varepsilon^{-b})_\varepsilon]A \subseteq U$ for all $b \le \log(\min_{i\in I_0} \eta_i) - \log(\max_{i\in I_0} C_i)$. $\square$

As in the classical theory of locally convex topological vector spaces an inspection of the neighborhoods of the origin gives some informations about "metrizability" and "normability".

Theorem 1.14. *Let $\mathcal{G}$ be a separated locally convex topological $\widetilde{\mathbb{C}}$-module with a countable base of neighborhoods of the origin. Then its topology is induced by a metric d invariant under translation.*

Proof. Let $(U_n)_{n\in\mathbb{N}}$ be a countable base of neighborhoods of the origin in $\mathcal{G}$. By Proposition 1.7 we may assume that each U_n is absorbent and absolutely convex. We define

$$f(u) = \sum_{n=0}^{\infty} 2^{-n} \min\{\mathcal{P}_{U_n}(u), 1\},$$

where $\mathcal{P}_{U_n}$ is the gauge of U_n. By Proposition 1.11 $f(u) = 0$ implies $u = 0$, and by construction we have that $f(u) = f(-u)$, $f(u+v) \le f(u) + f(v)$ for all $u, v \in \mathcal{G}$. At this point, as in the proof of Theorem 4 in [42, Chapter I] we obtain that $d(u, v) := f(u - v)$ is a distance invariant under translation which induces the original topology on $\mathcal{G}$. $\square$

The topology determined on a $\widetilde{\mathbb{C}}$-module $\mathcal{G}$ by an ultra-pseudo-norm $\mathcal{P}$ is a separated and locally convex $\widetilde{\mathbb{C}}$-linear topology such that every set $\{u \in \mathcal{G} : \mathcal{P}(u) \le \eta\}$ is bounded. This property characterizes the *ultra-pseudo-normed* $\widetilde{\mathbb{C}}$-modules.

Theorem 1.15. *If $\mathcal{G}$ is a separated locally convex topological $\widetilde{\mathbb{C}}$-module and it has a bounded neighborhood of the origin, then the topology on $\mathcal{G}$ is induced by an ultra-pseudo-norm.*

Proof. Let V be an absorbent and absolutely convex neighborhood of the origin contained in a bounded neighborhood of the origin. Then V is bounded, that is, for all neighborhoods U of the origin in $\mathcal{G}$ there exists $a \in \mathbb{R}$ such that $V \subseteq [(\varepsilon^b)_\varepsilon]U$ for $b \le a$. This means that $[(\varepsilon^{-b})_\varepsilon]V \subseteq U$ and that $\{[(\varepsilon^d)_\varepsilon]V\}_{d\in\mathbb{R}}$ is a base of the neighborhoods of the origin in $\mathcal{G}$. Therefore, the gauge $\mathcal{P}_V$ determines the topology on $\mathcal{G}$ which is separated. By Proposition 1.11, $\mathcal{P}_V$ is an ultra-pseudo-norm. $\square$

We conclude this subsection with some continuity issues.

Theorem 1.16. *Let $(\mathcal{G}, \{\mathcal{P}_i\}_{i\in I})$ be a locally convex topological $\widetilde{\mathbb{C}}$-module. An ultra-pseudo-seminorm $\mathcal{Q}$ on $\mathcal{G}$ is continuous if and only if it is continuous at the origin if and only if there exists a finite subset $I_0 \subseteq I$ and a constant $C > 0$ such that for all $u \in \mathcal{G}$*

$$\mathcal{Q}(u) \le C \max_{i\in I_0} \mathcal{P}_i(u). \tag{1.7}$$

Proof. Assume that $\mathcal{Q}$ is continuous at the origin and take $u_0 \in \mathcal{G}$, $u_0 \neq 0$. For all $\delta > 0$ there exists a finite subset $I_0 \subseteq I$ and $\eta > 0$ such that $\mathcal{Q}(u) \le \delta$ if $\max_{i\in I_0} \mathcal{P}_i(u) \le \eta$. Hence for all $u \in \mathcal{G}$ such that $\max_{i\in I_0} \mathcal{P}_i(u - u_0) \le \eta$ we have that $\mathcal{Q}(u - u_0) \le \delta$ and by definition of an ultra-pseudo-seminorm $|\mathcal{Q}(u) - \mathcal{Q}(u_0)| \le \mathcal{Q}(u - u_0)$. This shows that $\mathcal{Q}$ is continuous at $u_0 \in \mathcal{G}$.
It is clear that if $\mathcal{Q}$ satisfies (1.7) then it is continuous at the origin and consequently continuous on $\mathcal{G}$. Conversely if $\mathcal{Q}$ is continuous at the origin as before there exists a finite subset $I_0 \subseteq I$ and $\eta > 0$ such that $\max_{i\in I_0} \mathcal{P}_i(u) \le \eta$ implies $\mathcal{Q}(u) \le 1$. We begin by observing that $\mathcal{Q}(u) = 0$ when $\max_{i\in I_0} \mathcal{P}_i(u) = 0$. In fact if $\mathcal{P}_i(u) = 0$ for all $i \in I_0$ then

$$0 = |[(\varepsilon^b)_\varepsilon]|_{\mathrm{e}} \max_{i\in I_0} \mathcal{P}_i(u) = \max_{i\in I_0} \mathcal{P}_i([(\varepsilon^b)_\varepsilon]u)$$

and $\mathcal{Q}([(\varepsilon^b)_\varepsilon]u) = |[(\varepsilon^b)_\varepsilon]|_{\mathrm{e}}\mathcal{Q}(u) = \mathrm{e}^{-b}\mathcal{Q}(u) \le 1$ for all $b \in \mathbb{R}$. So when the ultra-pseudo-seminorm $\max_{i\in I_0} \mathcal{P}_i(u)$ is not zero we can write

$$\mathcal{Q}(v[(\varepsilon^a)_\varepsilon]) = \mathrm{e}^{-a}\,\mathcal{Q}(v), \tag{1.8}$$

where $a = \log(\eta / \max_{i\in I_0} \mathcal{P}_i(u))$, $v = u[(\varepsilon^{-a})_\varepsilon]$ and by construction $\mathcal{P}_i(v) = \mathrm{e}^a \mathcal{P}_i(u) \leq \eta$ for all $i \in I_0$. Combined with the continuity of $\mathcal{Q}$ at the origin, (1.8) leads to (1.7) and completes the proof. $\square$

Note that the composition of a $\widetilde{\mathbb{C}}$-linear map $T : \mathcal{G} \to \mathcal{H}$ between $\widetilde{\mathbb{C}}$-modules with an ultra-pseudo-seminorm on $\mathcal{H}$ gives an ultra-pseudo-seminorm on $\mathcal{G}$. Therefore, the following result concerning the continuity of $\widetilde{\mathbb{C}}$-linear maps between locally convex topological $\widetilde{\mathbb{C}}$-modules is a simple corollary of Theorem 1.16.

Corollary 1.17. *Let $(\mathcal{G}, \{\mathcal{P}_i\}_{i\in I})$ and $(\mathcal{H}, \{\mathcal{Q}_j\}_{j\in J})$ be locally convex topological $\widetilde{\mathbb{C}}$-modules. A $\widetilde{\mathbb{C}}$-linear map $T : \mathcal{G} \to \mathcal{H}$ is continuous if and only if it is continuous at the origin if and only if for all $j \in J$ there exists a finite subset $I_0 \subseteq I$ and a constant $C > 0$ such that for all $u \in \mathcal{G}$*

$$\mathcal{Q}_j(Tu) \leq C \max_{i\in I_0} \mathcal{P}_i(u). \tag{1.9}$$

1.3 Inductive limits and strict inductive limits of locally convex topological $\widetilde{\mathbb{C}}$-modules

In this subsection we consider a family of locally convex topological $\widetilde{\mathbb{C}}$-modules $(\mathcal{G}_\gamma)_{\gamma\in\Gamma}$ and the $\widetilde{\mathbb{C}}$-module of all the finite $\widetilde{\mathbb{C}}$-linear combinations of elements of $\cup_{\gamma\in\Gamma}\mathcal{G}_\gamma$, denoted by $\mathrm{span}(\cup_{\gamma\in\Gamma}\mathcal{G}_\gamma)$. We ask if the locally convex $\widetilde{\mathbb{C}}$-linear topologies τ_γ on $\mathcal{G}_\gamma$ can be pieced together to a locally convex $\widetilde{\mathbb{C}}$-linear topology τ on $\mathrm{span}(\cup_{\gamma\in\Gamma}\mathcal{G}_\gamma)$. More generally we can start from a given $\widetilde{\mathbb{C}}$-module $\mathcal{G}$ which is spanned by the images under some $\widetilde{\mathbb{C}}$-linear maps ι_γ of the original $\mathcal{G}_\gamma$'s.

Theorem 1.18. *Let $\mathcal{G}$ be a $\widetilde{\mathbb{C}}$-module, $(\mathcal{G}_\gamma)_{\gamma\in\Gamma}$ be a family of locally convex topological $\widetilde{\mathbb{C}}$-modules and $\iota_\gamma : \mathcal{G}_\gamma \to \mathcal{G}$ be a $\widetilde{\mathbb{C}}$-linear map so that $\mathcal{G} = \mathrm{span}(\cup_{\gamma\in\Gamma}\iota_\gamma(\mathcal{G}_\gamma))$. Let*

$$\mathcal{U} := \{U \subseteq \mathcal{G} \text{ absolutely convex} : \forall\gamma \in \Gamma,\ \iota_\gamma^{-1}(U) \text{ is a neighborhood of } 0 \text{ in } \mathcal{G}_\gamma\}.$$

The topology τ induced by the gauges $\{\mathcal{P}_U\}_{U\in\mathcal{U}}$ is the finest $\widetilde{\mathbb{C}}$-linear topology with a base of absolutely convex neighborhoods of the origin such that each ι_γ is continuous.

With this topology $\mathcal{G}$ is called an *inductive limit* of the locally convex topological $\widetilde{\mathbb{C}}$-modules $\mathcal{G}_\gamma$.

Proof. First we note that every $U \in \mathcal{U}$ is absorbent. In fact, $\iota_\gamma^{-1}(U)$ is an absorbent neighborhood of 0 in $\mathcal{G}_\gamma$ by Proposition 1.3 and then U absorbs every element of $\iota_\gamma(\mathcal{G}_\gamma)$. Now when we take $u_1 \in \mathcal{G}_{\gamma_1}$, $u_2 \in \mathcal{G}_{\gamma_2}$, $\iota_{\gamma_1}(u_1) + \iota_{\gamma_2}(u_2)$ is absorbed by U since we may write

$$\begin{aligned}\iota_{\gamma_1}(u_1) + \iota_{\gamma_2}(u_2) &\in [(\varepsilon^{b_1})_\varepsilon]U + [(\varepsilon^{b_2})_\varepsilon]U \\ &= [(\varepsilon^{\min\{b_1,b_2\}})_\varepsilon]([(\varepsilon^{b_1-\min\{b_1,b_2\}})_\varepsilon]U + [(\varepsilon^{b_2-\min\{b_1,b_2\}})_\varepsilon]U),\end{aligned}$$

for some $a_1, a_2 \in \mathbb{R}$ and for all $b_1 \le a_1$, $b_2 \le a_2$, where, as observed after Definition 1.1, $[(\varepsilon^{b_1-\min\{b_1,b_2\}})_\varepsilon]U + [(\varepsilon^{b_2-\min\{b_1,b_2\}})_\varepsilon]U$ is contained in U. This means that U is an absorbent subset of $\mathcal{G}$. By Proposition 1.9 and Theorem 1.10 the topology τ on $\mathcal{G}$ induced by the ultra-pseudo-seminorms $\{\mathcal{P}_U\}_{U\in\mathcal{U}}$ is a locally convex $\widetilde{\mathbb{C}}$-linear topology i.e. a $\widetilde{\mathbb{C}}$-linear topology with a base of absolutely convex neighborhoods of the origin. By definition of τ it is clear that every $\iota_\gamma : \mathcal{G}_\gamma \to (\mathcal{G}, \tau)$ is continuous. Assume now that τ' is another locally convex $\widetilde{\mathbb{C}}$-linear topology on $\mathcal{G}$ which makes each ι_γ continuous. τ is finer than τ' because if U' is an absolutely convex neighborhood of 0 for τ' then $\iota_\gamma^{-1}(U)$ is a neighborhood of 0 in $\mathcal{G}_\gamma$ for all $\gamma \in \Gamma$. $\square$

Continuity of $\widetilde{\mathbb{C}}$-linear maps between locally convex topological $\widetilde{\mathbb{C}}$-modules $\mathcal{G}$ and $\mathcal{H}$ can easily be described when $\mathcal{G}$ has an inductive limit topology.

Proposition 1.19. *Let $\mathcal{G}$ be the inductive limit of the locally convex topological $\widetilde{\mathbb{C}}$-modules $(\mathcal{G}_\gamma)_{\gamma\in\Gamma}$ and $\mathcal{H}$ be a locally convex topological $\widetilde{\mathbb{C}}$-module. A $\widetilde{\mathbb{C}}$-linear map $T : \mathcal{G} \to \mathcal{H}$ is continuous if and only if for each $\gamma \in \Gamma$ the composition $T \circ \iota_\gamma : \mathcal{G}_\gamma \to \mathcal{H}$ is continuous.*

Proof. The non-trivial assertion to prove is that T is continuous if every $T \circ \iota_\gamma$ is continuous. By continuity at 0, for every neighborhood V of the origin in $\mathcal{H}$, $\iota_\gamma^{-1}(T^{-1}(V))$ is a neighborhood of 0 in $\mathcal{G}_\gamma$. Since we may choose V absolutely convex and the $\widetilde{\mathbb{C}}$-linearity of T guarantees that $T^{-1}(V)$ itself is absolutely convex in $\mathcal{G}$, the proof is complete. $\square$

Definition 1.20. *Let $\mathcal{G}$ be a $\widetilde{\mathbb{C}}$-module and $(\mathcal{G}_n)_{n\in\mathbb{N}}$ be a sequence of $\widetilde{\mathbb{C}}$-submodules of $\mathcal{G}$ such that $\mathcal{G}_n \subseteq \mathcal{G}_{n+1}$ for all $n \in \mathbb{N}$ and $\mathcal{G} = \cup_{n\in\mathbb{N}}\mathcal{G}_n$. Assume that $\mathcal{G}_n$ is equipped with a locally convex $\widetilde{\mathbb{C}}$-linear topology τ_n such that the topology induced by τ_{n+1} on $\mathcal{G}_n$ is τ_n.*
Then $\mathcal{G}$ endowed the inductive limit topology τ is called the strict inductive limit *of the sequence $(\mathcal{G}_n)_{n\in\mathbb{N}}$ of locally convex topological $\widetilde{\mathbb{C}}$-modules.*

Proposition 1.21. *Let $\mathcal{G}$ be the strict inductive limit of the sequence of locally convex topological $\widetilde{\mathbb{C}}$-modules $(\mathcal{G}_n, \tau_n)_{n\in\mathbb{N}}$. The topology τ on $\mathcal{G}$ induces the original topology τ_n on each $\mathcal{G}_n$.*

The proof of this proposition requires a technical lemma.

Lemma 1.22. *Let $\mathcal{G}$ be a locally convex topological $\widetilde{\mathbb{C}}$-module. Let M be a $\widetilde{\mathbb{C}}$-submodule of $\mathcal{G}$ and V be a convex neighborhood of the origin in M. Then there exists a convex neighborhood W of 0 in $\mathcal{G}$ such that $W \cap M = V$.*

Proof. By definition of the induced topology on M there exists a convex neighborhood U of 0 in $\mathcal{G}$ such that $U \cap M \subseteq V$. Let $W = U + V$. W is the convex hull of $U \cup V$ since it can be written as $\{\sum_{i=1}^n [(\varepsilon^{b_i})_\varepsilon]u_i,\ n \in \mathbb{N},\ b_i \ge 0,\ u_i \in U \cup V\}$, recalling the considerations after Definition 1.1. From $U \subseteq W$ we have that W is a convex neighborhood of 0 in $\mathcal{G}$ such that $V \subseteq W \cap M$. It remains to prove the opposite inclusion. First, $w \in W$ is of the form $w = u+v$ for some $u \in U$ and $v \in V$. Therefore, if $w \in M$ we have that $u = w - v \in U \cap M$. Since $U \cap M \subseteq V$ we conclude that u is an element of V. This leads to $W \cap M \subseteq V$. $\square$

Remark 1.23. Note that if U and V are both convex and balanced then $U+V$ is the absolutely convex hull of $U \cup V$ i.e. the set of all finite sums $\sum_{i=1}^{n} \lambda_i u_i$ where $u_i \in U \cup V$, $\lambda_i \in \widetilde{\mathbb{C}}$ and $\max_{i=1,\dots,n} |\lambda_i|_{\mathrm{e}} \le 1$.

Proof of Proposition 1.21. Denoting the topology induced by τ on $\mathcal{G}_n$ by τ_n', it is clear that τ_n' is coarser than τ_n. It remains to prove that any absolutely convex neighborhood V_n of the origin in the topology τ_n is obtained as the intersection of a neighborhood of 0 in $\mathcal{G}$ with the $\widetilde{\mathbb{C}}$-module $\mathcal{G}_n$. Lemma 1.22 and Remark 1.23 allow us to construct a sequence $(V_{n+p})_{p\in\mathbb{N}}$ such that V_{n+p} is an absolutely convex neighborhood of the origin in $\mathcal{G}_{n+p}$ for τ_{n+p}, $V_{n+p} \subseteq V_{n+p+1}$ and $V_{n+p} \cap \mathcal{G}_n = V_n$ for all p. In conclusion, $V = \cup_{p\in\mathbb{N}} V_{n+p}$ is a neighborhood of the origin in $\mathcal{G}$ such that $V \cap \mathcal{G}_n = V_n$. □

The following statements concerning separated $\widetilde{\mathbb{C}}$-modules and the closedness of $\mathcal{G}_n$ in $\mathcal{G}$ are immediate consequences of Proposition 1.21. We refer to [28, Chapter 2, Section 12, Cor. 1,2] for a proof.

Corollary 1.24. *Under the hypotheses of Proposition 1.21, $\mathcal{G}$ is separated if each $\mathcal{G}_n$ is separated.*

Corollary 1.25. *Under the hypotheses of Proposition 1.21, if each $\mathcal{G}_n$ is closed in $\mathcal{G}_{n+1}$ for the topology τ_{n+1} then $\mathcal{G}_n$ is closed in $\mathcal{G}$ for τ.*

We conclude the collection of results involving strict inductive limits of locally convex topological $\widetilde{\mathbb{C}}$-modules by characterizing bounded subsets.

Theorem 1.26. *Let $(\mathcal{G}, \tau)$ be the strict inductive limit of the sequence of locally convex topological $\widetilde{\mathbb{C}}$-modules $(\mathcal{G}_n, \tau_n)_{n\in\mathbb{N}}$. Assume in addition that each $\mathcal{G}_n$ is closed in $\mathcal{G}_{n+1}$ with respect to τ_{n+1}. Then $A \subseteq \mathcal{G}$ is bounded if and only if A is contained in some $\mathcal{G}_n$ and bounded there.*

The proof of Theorem 1.26 requires some preliminary lemmas.

Lemma 1.27. *A set A in a topological $\widetilde{\mathbb{C}}$-module $\mathcal{G}$ is bounded if and only if for all sequences $(u_n)_n$ of elements of A and all sequences $(\lambda_n)_n$ in $\widetilde{\mathbb{C}}$ converging to 0, the sequence $(\lambda_n u_n)_n$ tends to 0 in $\mathcal{G}$.*

Proof. Let A be a bounded subset of $\mathcal{G}$ and V be a balanced neighborhood of the origin. Since $A \subseteq [(\varepsilon^a)_\varepsilon]V$ for some $a \in \mathbb{R}$ we have that $\mathcal{P}_V(u) \le \mathrm{e}^{-a}$ on A. As shown in the proof of Proposition 1.9, the estimate $\mathcal{P}_V(\lambda u) \le |\lambda|_{\mathrm{e}} \mathcal{P}_V(u)$ holds for all $\lambda \in \widetilde{\mathbb{C}}$ and $u \in \mathcal{G}$. Therefore $\mathcal{P}_V(\lambda_n u_n) \le |\lambda_n|_{\mathrm{e}} \mathcal{P}_V(u_n) \le |\lambda_n|_{\mathrm{e}} \mathrm{e}^{-a}$ and $\lambda_n \to 0$ in $\widetilde{\mathbb{C}}$ yields $\mathcal{P}_V(\lambda_n u_n) < 1$ for n larger than some $N \in \mathbb{N}$. As a consequence, $\lambda_n u_n \in V$ if $n \ge N$ and $\lambda_n u_n$ is convergent to 0 in $\mathcal{G}$.
Suppose now that all the sequences $(\lambda_n u_n)_n$, where $(u_n)_n \subseteq A$ and $\lambda_n \to 0$ in $\widetilde{\mathbb{C}}$, tend to 0 in $\mathcal{G}$. Then A is necessarily bounded. In fact if A is not bounded there exists a balanced neighborhood of the origin U and a sequence $b_n \to -\infty$ such that $A \cap (\mathcal{G} \setminus [(\varepsilon^{b_n})_\varepsilon]U) \neq \emptyset$. Choosing $u_n \in A \cap (\mathcal{G} \setminus [(\varepsilon^{b_n})_\varepsilon]U)$, the sequence $[(\varepsilon^{-b_n})_\varepsilon]$ goes to 0 in $\widetilde{\mathbb{C}}$ but $[(\varepsilon^{-b_n})_\varepsilon]u_n$ is not convergent to 0 in $\mathcal{G}$ since $[(\varepsilon^{-b_n})_\varepsilon]u_n \notin U$ for all $n \in \mathbb{N}$. This contradicts our hypothesis. □

Lemma 1.28. *Under the assumptions of Lemma 1.22, if M is closed then for every $u_0 \notin M$ there exists a convex neighborhood W_0 of 0 in $\mathcal{G}$ such that $W_0 \cap M = V$ and $u_0 \notin W_0$.*

Proof. If M is closed then by Remark 1.4 and Example 1.12 $\mathcal{G}/M$ is a separated locally convex topological $\widetilde{\mathbb{C}}$-module. This implies that there exists a convex neighborhood U_0 of 0 in $\mathcal{G}$ such that $[u_0] \notin \pi(U_0)$. Hence $(u_0 + M) \cap U_0 = \emptyset$. By Lemma 1.22 there exists a convex neighborhood W of 0 in $\mathcal{G}$ such that $W \cap M = V$. Therefore taking $W \cap U_0$, we can state that there exists a convex neighborhood U_0' of 0 in $\mathcal{G}$ such that $(u_0 + M) \cap U_0' = \emptyset$ and $U_0' \cap M \subseteq V$. The same reasoning as in Lemma 1.22 combined with $(u_0 + M) \cap U_0' = \emptyset$ shows that $W_0 = U_0' + V$ is a convex neighborhood of 0 in $\mathcal{G}$ such that $W_0 \cap M = V$ and $u_0 \notin W_0$. $\square$

Note that if we choose V and U_0 absolutely convex then by Remark 1.23 we obtain that W_0 is an absolutely convex neighborhood of the origin in $\mathcal{G}$.

Proof of Theorem 1.26. If $A \subseteq \mathcal{G}_n$ is bounded for the topology τ_n then the continuity of the embedding of $(\mathcal{G}_n, \tau_n)$ into $(\mathcal{G}, \tau)$ guarantees that A is bounded in $\mathcal{G}$.
Suppose now that A is not contained in any $\widetilde{\mathbb{C}}$-module $\mathcal{G}_n$ and choose a sequence of elements $u_n \in A \cap (\mathcal{G} \setminus \mathcal{G}_n)$. There exists a strictly increasing sequence $(n_k)_k$ of natural numbers and a subsequence $(v_k)_k$ of $(u_n)_n$ such that $v_k \in \mathcal{G}_{n_{k+1}} \setminus \mathcal{G}_{n_k}$. By Lemma 1.28 we can construct an increasing sequence $(V_k)_k$ of absolutely convex sets such that V_k is a neighborhood of 0 in $\mathcal{G}_{n_k}$, $V_{k+1} \cap \mathcal{G}_{n_k} = V_k$ and $[(\varepsilon^k)_\varepsilon] v_k \notin V_{k+1}$. As in the proof of Proposition 1.21, $V = \cup_{k \in \mathbb{N}} V_k$ is a neighborhood of the origin in $\mathcal{G}$ which does not contain $[(\varepsilon^k)_\varepsilon] v_k$ for any $k \in \mathbb{N}$. Then $[(\varepsilon^k)_\varepsilon] \to 0$ in $\widetilde{\mathbb{C}}$ but the sequence $([(\varepsilon^k)_\varepsilon] v_k)_k$ is not convergent to 0 in $\mathcal{G}$. By Lemma 1.27 it follows that A cannot be bounded in $\mathcal{G}$.
Finally, by Proposition 1.21 it is clear that if A is contained in some $\mathcal{G}_n$ and bounded in $\mathcal{G}$ it has to be bounded in $\mathcal{G}_n$ as well. $\square$

Every sequence $(u_n)_n$ in $\mathcal{G}$ which is tending to 0 is an example of bounded set in $\mathcal{G}$. In fact for each absolutely convex neighborhood U of the origin, there exists $N \in \mathbb{N}$ such that $u_n \in U$ for all $n \geq N$, and noting that $[(\varepsilon^{b_1})_\varepsilon] U \subseteq [(\varepsilon^{b_2})_\varepsilon] U$ if $b_1 \geq b_2$, there exists $a \in \mathbb{R}$ such that $u_n \in [(\varepsilon^b)_\varepsilon] U$ for all $n \in \mathbb{N}$ and $b \leq a$. At this point recalling Proposition 1.21 it is immediate to prove the following corollary of Theorem 1.26.

Corollary 1.29. *Under the assumptions of Theorem 1.26 a sequence $(u_n)_n$ is convergent to 0 in $\mathcal{G}$ if and only if it is contained in some $\mathcal{G}_n$ and convergent to 0 there.*

1.4 Completeness

In this subsection we adapt the theory of complete topological vector spaces [28] to the context of $\widetilde{\mathbb{C}}$-modules. We say that a subset A of a topological $\widetilde{\mathbb{C}}$-module is *complete* if every Cauchy filter on A converges to some point of A and that

a topological $\widetilde{\mathbb{C}}$-module $\mathcal{G}$ is *quasi-complete* if every bounded closed subset is complete.

Remark 1.30. A topological $\widetilde{\mathbb{C}}$-module $\mathcal{G}$ is a uniform space [5, 49] and hence the following properties hold which will be used repeatedly later.

Let A be a subset of $\mathcal{G}$. Any filter $\mathcal{F}$ on A convergent to some $u \in \mathcal{G}$ is a Cauchy filter and if $u \in \mathcal{G}$ adheres to a Cauchy filter $\mathcal{O}$ on A then $\mathcal{O}$ converges to u. Moreover a complete subset of a separated topological $\widetilde{\mathbb{C}}$-module is closed and if $A \subseteq \mathcal{G}$ is complete every closed subset of A is complete itself. Finally in a metrizable topological $\widetilde{\mathbb{C}}$-module $\mathcal{G}$ a subset A is complete if and only if every Cauchy sequence of points of A converges to some point of A.

Note that even if $\mathcal{G}$ is only quasi-complete, every Cauchy sequence $(u_n)_n \subseteq \mathcal{G}$ is convergent. First of all since $(u_n)_n$ is a Cauchy sequence the set $U := \{u_n,\ n \in \mathbb{N}\}$ is bounded in $\mathcal{G}$. We recall that for all neighborhoods V of 0 in a topological $\widetilde{\mathbb{C}}$-module we can find a balanced neighborhood W of 0 such that $W + W \subseteq V$. This means that $\overline{W} \subseteq V$ and therefore the closure of a bounded subset of a topological $\widetilde{\mathbb{C}}$-module is still bounded. Then in our case $\overline{U}$ is closed and bounded in $\mathcal{G}$ and by the quasi-completeness of $\mathcal{G}$ the sequence $(u_n)_n \subseteq \overline{U}$ is convergent.

A locally convex topological $\widetilde{\mathbb{C}}$-module which is metrizable and complete is called a *Fréchet* $\widetilde{\mathbb{C}}$*-module*. As a straightforward application of Remark 1.30 we show that $\widetilde{\mathbb{C}}$ is complete. The proof of this result is essentially due to Scarpalézos [47, Proposition 2.1].

Proposition 1.31. $\widetilde{\mathbb{C}}$ *with the topology given by the ultra-pseudo-norm* $|\cdot|_{\mathrm{e}}$ *is complete.*

Proof. By Remark 1.30 it is sufficient to prove that every Cauchy sequence $(u_n)_n$ in $\widetilde{\mathbb{C}}$ is convergent. We know that for every $\eta > 0$ there exists $N \in \mathbb{N}$ such that for all $m, p \geq N$, $|u_m - u_p|_{\mathrm{e}} \leq \eta$. Considering representatives and the valuation on $\widetilde{\mathbb{C}}$ defined via $\mathrm{v} : \mathcal{E}_M \to (-\infty, +\infty]$ in (1.1), we can extract a subsequence $(u_{n_k})_k$ such that $\mathrm{v}((u_{n_{k+1},\varepsilon} - u_{n_k,\varepsilon})_\varepsilon) > k$ for all $k \in \mathbb{N}$. This means that we can find $\varepsilon_k \searrow 0$, $\varepsilon_k \leq 1/2^k$ such that $|u_{n_{k+1},\varepsilon} - u_{n_k,\varepsilon}| \leq \varepsilon^k$ on $(0, \varepsilon_k)$. Let

$$h_{k,\varepsilon} = \begin{cases} u_{n_{k+1},\varepsilon} - u_{n_k,\varepsilon} & \varepsilon \in (0, \varepsilon_k), \\ 0 & \varepsilon \in [\varepsilon_k, 1]. \end{cases}$$

$(h_{k,\varepsilon})_\varepsilon \in \mathcal{E}_M$ since $|h_{k,\varepsilon}| \leq \varepsilon^k$ on the interval $(0,1]$. Moreover the sum $u_\varepsilon := u_{n_0,\varepsilon} + \sum_{k=0}^{\infty} h_{k,\varepsilon}$ is locally finite and by

$$|u_\varepsilon| \leq |u_{n_0,\varepsilon}| + \sum_{k=0}^{\infty} |h_{k,\varepsilon}| \leq |u_{n_0,\varepsilon}| + \sum_{k=0}^{\infty} \frac{1}{2^k}$$

it defines the representative of a complex generalized number $u = [(u_\varepsilon)_\varepsilon]$. The sequence u_{n_k} tends to u in $\widetilde{\mathbb{C}}$. In fact, for all $\overline{k} \geq 1$ the estimate

$$|u_{n_{\overline{k}},\varepsilon} - u_\varepsilon| = \Big| - \sum_{k=\overline{k}}^{\infty} h_{k,\varepsilon}\Big| \leq \sum_{k=\overline{k}}^{\infty} \varepsilon^{k-1}\varepsilon_k \leq \varepsilon^{\overline{k}-1} \sum_{k=\overline{k}}^{\infty} \frac{1}{2^k},$$

valid on the interval $(0, \varepsilon_{\overline{k}-1})$, yields $\mathrm{v}((u_{n_k,\varepsilon} - u_\varepsilon)_\varepsilon) \to +\infty$. Thus $(u_n)_n$ is a Cauchy sequence with a convergent subsequence and it converges to the same point $u \in \widetilde{\mathbb{C}}$. □

It is possible to decide if a strict inductive limit of locally convex topological $\widetilde{\mathbb{C}}$-modules is complete by looking at the terms $\mathcal{G}_n$ of the sequence which defines it.

Theorem 1.32. *Let $(\mathcal{G}, \tau)$ be the strict inductive limit of the sequence of locally convex topological $\widetilde{\mathbb{C}}$-modules $(\mathcal{G}_n, \tau_n)_{n\in\mathbb{N}}$ where $\mathcal{G}_n$ is assumed to be closed in $\mathcal{G}_{n+1}$ for τ_{n+1}. Then $\mathcal{G}$ is complete if and only if all the $\mathcal{G}_n$ are complete.*

Before proving this theorem we present a technical lemma which will turn out to be useful later on as well.

Lemma 1.33. *Let $\mathcal{F}$ be a Cauchy filter on the strict inductive limit $(\mathcal{G}, \tau)$ of Theorem 1.32 and $\mathcal{O}$ be the Cauchy filter whose base is formed by all the sets $M + V$ where M runs through $\mathcal{F}$ and V through the filter of neighborhoods of the origin in $\mathcal{G}$. Then there exists an integer n such that $\mathcal{O}$ induces a Cauchy filter on $\mathcal{G}_n$.*

Proof. If there exists $n \in \mathbb{N}$ such that $(M + V) \cap \mathcal{G}_n \neq \emptyset$ for all $M \in \mathcal{F}$ and neighborhoods V of the origin in $\mathcal{G}$ then by Proposition 1.21 the lemma is proven. We assume therefore that this is not the case, i.e. that for all $n \in \mathbb{N}$ there exist $M_n \in \mathcal{F}$ and a neighborhood V_n of the origin in $\mathcal{G}$ such that $(M_n + V_n) \cap \mathcal{G}_n = \emptyset$. In addition we may assume that $M_n - M_n \subseteq V_n$ and that $(V_n)_n$ is a decreasing sequence of absolutely convex neighborhoods. Consider the absolutely convex hull W of $\cup_{n\in\mathbb{N}}(V_n \cap \mathcal{G}_n)$. Since every V_n is absolutely convex it coincides with the set of all finite sums of elements of $\cup_{n\in\mathbb{N}}(V_n \cap \mathcal{G}_n)$ and by construction it is a neighborhood of the origin in $\mathcal{G}$. We want to prove that no $Q \in \mathcal{F}$ has the property $Q - Q \subseteq W$. For this purpose we take $W_n := V_0 \cap \mathcal{G}_0 + V_1 \cap \mathcal{G}_1 + \cdots + V_{n-1} \cap \mathcal{G}_{n-1} + V_n$ which is the absolutely convex hull of $(\cup_{i\le n-1} V_i \cap \mathcal{G}_i) \cup V_n$. W_n is a neighborhood of the origin in $\mathcal{G}$ and $W \subseteq W_n$ for all n. Since $\mathcal{F}$ is a Cauchy filter there exists $P_n \in \mathcal{F}$ such that $P_n - P_n \subseteq W_n$. This implies $(P_n + W_n) \cap \mathcal{G}_n = \emptyset$. In fact for $u_0 \in P_n \cap M_n$ we have that $P_n \subseteq u_0 + W_n$ and as a consequence every element y of P_n has the form $y = u_0 + \sum_{i=0}^{n} v_i$ where $v_i \in V_i \cap \mathcal{G}_i$ if $i = 0, ..., n-1$ and $v_n \in V_n$. At this point $z \in P_n + W_n$ may be written as $z = u_0 + \sum_{i=0}^{n-1}(v_i + v'_i) + v_n + v'_n$ with $v_i, v'_i \in V_i \cap \mathcal{G}_n$ for $i = 0, ..., n-1$ and $v_n, v'_n \in V_n$. Note that since V_n is convex $v_n + v'_n \in V_n$ and that $\sum_{i=0}^{n-1}(v_i + v'_i) \in \mathcal{G}_n$. By $(M_n + V_n) \cap \mathcal{G}_n = \emptyset$ it follows that $z \notin \mathcal{G}_n$.
Finally suppose that there exists $Q \in \mathcal{F}$ such that $Q - Q \subseteq W$ and that $y_0 \in Q$. Then $y_0 \in \mathcal{G}_n$ for some n and $Q \cap P_n = \emptyset$ which contradicts the hypothesis that $\mathcal{F}$ is a filter. Indeed by construction of P_n and W_n if $y \in P_n$ then $y_0 - y \in {W_n}^{\mathrm{c}} \subseteq W^{\mathrm{c}}$. Hence y does not belong to Q. □

Proof of Theorem 1.32. If $\mathcal{G}$ is complete, recalling that by Corollary 1.25 every $\mathcal{G}_n$ is closed in $(\mathcal{G}, \tau)$, by Remark 1.30 every $\mathcal{G}_n$ is complete. Conversely assume that each $(\mathcal{G}_n, \tau_n)$ is complete and take a Cauchy filter $\mathcal{F}$ on $\mathcal{G}$. The filter $\mathcal{O}$ constructed in Lemma 1.33 induces a Cauchy filter $\mathcal{O}_n$ on some $\mathcal{G}_n$. Hence $\mathcal{O}_n$

converges to some $u \in \mathcal{G}_n$ and, since by Proposition 1.21 τ_n is the topology induced by τ on $\mathcal{G}_n$, u adheres to the filter $\mathcal{O}$. Consequently $\mathcal{O}$ converges to u and the same conclusion holds for $\mathcal{F}$, since it is finer than $\mathcal{O}$. $\square$

We finally consider a family of topological $\widetilde{\mathbb{C}}$-modules $(\mathcal{G}_\gamma)_{\gamma\in\Gamma}$ and a $\widetilde{\mathbb{C}}$-module $\mathcal{G}$ such that for each $\gamma \in \Gamma$ there exists a $\widetilde{\mathbb{C}}$-linear map $\iota_\gamma : \mathcal{G} \to \mathcal{G}_\gamma$. The *initial topology* on $\mathcal{G}$ is the coarsest topology such that each ι_γ is continuous. By the $\widetilde{\mathbb{C}}$-linearity of ι_γ we have that such a topology is $\widetilde{\mathbb{C}}$-linear and a base of neighborhoods of the origin is given by all the finite intersections $\iota_{\gamma_1}^{-1}(U_1) \cap \iota_{\gamma_2}^{-1}(U_2) ... \cap \iota_{\gamma_n}^{-1}(U_n)$ where U_i, $i = 1, 2, ..., n$, is a neighborhood of 0 in $\mathcal{G}_{\gamma_i}$. In particular if the $\mathcal{G}_\gamma$ are locally convex topological $\widetilde{\mathbb{C}}$-modules with ultra-pseudo-seminorms $\{\mathcal{P}_{j,\gamma}\}_{j\in J_\gamma}$ then the initial topology on $\mathcal{G}$ is determined by the family of ultra-pseudo-seminorms $\{\mathcal{P}_{j,\gamma} \circ \iota_\gamma\}_{j\in J_\gamma,\gamma\in\Gamma}$. Let now I be an ordered set of indices and $(\mathcal{G}_i)_{i\in I}$ be a family of topological $\widetilde{\mathbb{C}}$-modules such that $\mathcal{G}_j \subseteq \mathcal{G}_i$ if $j \geq i$. The intersection $\mathcal{G} := \cap_{i\in I}\mathcal{G}_i$ is naturally endowed with the initial topology defined by $(\mathcal{G}_i)_{i\in I}$ and the injections $\mathcal{G} \to \mathcal{G}_i$. Adapting the reasoning of Proposition 3 and the corresponding corollary in [28, Chapter 2, Section 11] to the context of topological $\widetilde{\mathbb{C}}$-modules, we prove that the completeness of each $\mathcal{G}_i$ may be transferred to the intersection $\mathcal{G}$ under suitable hypotheses.

Proposition 1.34. *Let $(\mathcal{G}_i)_{i\in I}$ be a family of separated topological $\widetilde{\mathbb{C}}$-modules where the index set is ordered. Suppose that for $i \leq j$, $\mathcal{G}_j$ is a $\widetilde{\mathbb{C}}$-submodule of $\mathcal{G}_i$ and the topology on $\mathcal{G}_j$ is finer than the topology induced by $\mathcal{G}_i$ on $\mathcal{G}_j$. Let $\mathcal{G} = \cap_{i\in I}\mathcal{G}_i$ be equipped with the initial topology for the injections $\mathcal{G} \to \mathcal{G}_i$. If the $\mathcal{G}_i$ are complete then $\mathcal{G}$ is complete.*

2 Duality theory for topological $\widetilde{\mathbb{C}}$-modules

This section is devoted to the dual of a topological $\widetilde{\mathbb{C}}$-module $\mathcal{G}$ i.e. the $\widetilde{\mathbb{C}}$-module $\mathrm{L}(\mathcal{G}, \widetilde{\mathbb{C}})$ of all $\widetilde{\mathbb{C}}$-linear and continuous maps on $\mathcal{G}$ with values in $\widetilde{\mathbb{C}}$. We present different ways of endowing $\mathrm{L}(\mathcal{G}, \widetilde{\mathbb{C}})$ with a $\widetilde{\mathbb{C}}$-linear topology and deal with related topics as pairings of $\widetilde{\mathbb{C}}$-modules, weak topologies, polar sets and polar topologies.

Definition 2.1. *Let $\mathcal{G}$ and $\mathcal{H}$ be two $\widetilde{\mathbb{C}}$-modules. If a $\widetilde{\mathbb{C}}$-bilinear form b : $\mathcal{G} \times \mathcal{H} \to \widetilde{\mathbb{C}} : (u, v) \to \mathsf{b}(u, v)$ is given we say that $\mathcal{G}$ and $\mathcal{H}$ form a pairing with respect to b. The pairing separates points of $\mathcal{G}$ if for all $u \neq 0$ in $\mathcal{G}$ there exists $v \in \mathcal{H}$ such that $\mathsf{b}(u, v) \neq 0$. Analogously it separates points of $\mathcal{H}$ if for all $v \neq 0$ in $\mathcal{H}$ there exists $u \in \mathcal{G}$ such that $\mathsf{b}(u, v) \neq 0$. The pairing is separated if it separates points of both $\mathcal{G}$ and $\mathcal{H}$.*

A pairing $(\mathcal{G}, \mathcal{H}, \mathsf{b})$ defines a topology on each involved $\widetilde{\mathbb{C}}$-module via the $\widetilde{\mathbb{C}}$-bilinear form b. The *weak topology* on $\mathcal{G}$ is the coarsest topology $\sigma(\mathcal{G}, \mathcal{H})$ on $\mathcal{G}$ such that each map $\mathsf{b}(\cdot, v) : \mathcal{G} \to \widetilde{\mathbb{C}} : u \to \mathsf{b}(u, v)$, for v varying in $\mathcal{H}$, is continuous. Every $\mathsf{b}(\cdot, v)$ is $\widetilde{\mathbb{C}}$-linear and continuous if and only if the ultra-pseudo-seminorm $\mathcal{P}_v : \mathcal{G} \to [0, \infty) : u \to |\mathsf{b}(u, v)|_\mathrm{e}$ is continuous. Hence $\sigma(\mathcal{G}, \mathcal{H})$ is the topology induced by the family of ultra-pseudo-seminorms $\{\mathcal{P}_v\}_{v\in\mathcal{H}}$ and

by Theorem 1.10 it provides the structure of a locally convex topological $\widetilde{\mathbb{C}}$-module on $\mathcal{G}$. Obviously the same holds for $(\mathcal{H}, \sigma(\mathcal{H}, \mathcal{G}))$.

Combining Definition 2.1 with Proposition 1.11 we obtain that the pairing $(\mathcal{G}, \mathcal{H}, \mathsf{b})$ separates points of $\mathcal{G}$ if and only if $\sigma(\mathcal{G}, \mathcal{H})$ is a Hausdorff topology. Any $\widetilde{\mathbb{C}}$-module $\mathcal{G}$ with its algebraic dual $L(\mathcal{G}, \widetilde{\mathbb{C}})$ and any topological $\widetilde{\mathbb{C}}$-module $\mathcal{G}$ with its topological dual $\L(\mathcal{G}, \widetilde{\mathbb{C}})$ forms a pairing via the canonical $\widetilde{\mathbb{C}}$-bilinear map $\langle u, T\rangle = T(u)$. By the previous considerations the topologies $\sigma(L(\mathcal{G}, \widetilde{\mathbb{C}}), \mathcal{G})$ and $\sigma(\L(\mathcal{G}, \widetilde{\mathbb{C}}), \mathcal{G})$ are separated.

Proposition 2.2. *Let $\mathcal{G}$ be a $\widetilde{\mathbb{C}}$-module. $L(\mathcal{G}, \widetilde{\mathbb{C}})$ is complete for the weak topology $\sigma(L(\mathcal{G}, \widetilde{\mathbb{C}}), \mathcal{G})$.*

Proof. Let $\mathcal{F}$ be a Cauchy filter on $L(\mathcal{G}, \widetilde{\mathbb{C}})$. For all $u \in \mathcal{G}$, $\mathcal{F}_u$, the filter having as a base the family $\{X_u\}_{X\in\mathcal{F}}$ where $X_u := \{T(u) : T \in X\}$, is a Cauchy filter on $\widetilde{\mathbb{C}}$. Since $\widetilde{\mathbb{C}}$ is complete, $\mathcal{F}_u$ is convergent to some $F(u) \in \widetilde{\mathbb{C}}$. An easy adaptation of the proof of Proposition 13 in [42, Chapter III, Section 6] to the $\widetilde{\mathbb{C}}$-module $\mathcal{G}$ and its algebraic dual $L(\mathcal{G}, \widetilde{\mathbb{C}})$ shows that $F : u \to F(u)$ is $\widetilde{\mathbb{C}}$-linear and that $\mathcal{F} \to F$ according to the weak topology $\sigma(L(\mathcal{G}, \widetilde{\mathbb{C}}), \mathcal{G})$. $\square$

Definition 2.3. *Let $(\mathcal{G}, \mathcal{H}, \mathsf{b})$ be a pairing of $\widetilde{\mathbb{C}}$-modules and A be a subset of $\mathcal{G}$. The* polar *of A is the subset A° of $\mathcal{H}$ of those $v \in \mathcal{H}$ such that $|\mathsf{b}(u, v)|_{\mathrm{e}} \le 1$ for all $u \in A$. Similarly we define the polar of a subset of $\mathcal{H}$.*

Some elementary properties of polar sets are collected in the following proposition.

Proposition 2.4.

(i) If $A_1 \subseteq A_2$ then $A_2^\circ \subseteq A_1^\circ$.

(ii) The polar set of $A \subseteq \mathcal{G}$ is a balanced convex subset of $\mathcal{H}$ closed for $\sigma(\mathcal{H}, \mathcal{G})$.

(iii) For all invertible $\lambda \in \widetilde{\mathbb{C}}$, $(\lambda A)^\circ = \lambda^{-1}A^\circ$. In particular A° is absorbent if and only if A is bounded in $(\mathcal{G}, \sigma(\mathcal{G}, \mathcal{H}))$.

(iv) $(\bigcup_{i\in I} A_i)^\circ = \bigcap_{i\in I} A_i^\circ$.

Proof. We omit the proof of the first and the fourth assertion since it is a simple application of Definition 2.3.

Let $A \subseteq \mathcal{G}$. A° is balanced since for all $\lambda \in \widetilde{\mathbb{C}}$ with $|\lambda|_{\mathrm{e}} \le 1$, if $v \in A^\circ$ then $|\mathsf{b}(u, \lambda v)|_{\mathrm{e}} = |\lambda \mathsf{b}(u, v)|_{\mathrm{e}} \le |\lambda|_{\mathrm{e}} |\mathsf{b}(u, v)|_{\mathrm{e}} \le 1$ on A. For each $v_1, v_2 \in A^\circ$ and $u \in A$, the estimate

$$|\mathsf{b}(u, v_1 + v_2)|_{\mathrm{e}} = |\mathsf{b}(u, v_1) + \mathsf{b}(u, v_2)|_{\mathrm{e}} \le \max\{|\mathsf{b}(u, v_1)|_{\mathrm{e}}, |\mathsf{b}(u, v_2)|_{\mathrm{e}}\} \le 1$$

holds, i.e. $A^\circ + A^\circ \subseteq A^\circ$. This result combined with the fact that A° is balanced shows that A° is convex. Finally A° is closed in $(\mathcal{H}, \sigma(\mathcal{H}, \mathcal{G}))$ since it may be written as $\cap_{u\in A} A_u$ where $A_u := \{v \in \mathcal{H} : |\mathsf{b}(u, v)|_{\mathrm{e}} \le 1\}$ is closed by definition of the weak topology on $\mathcal{H}$.

Take now λ invertible in $\widetilde{\mathbb{C}}$. The equality $(\lambda A)^\circ = \lambda^{-1}A^\circ$ is guaranteed by the $\widetilde{\mathbb{C}}$-bilinearity of b. If A° is absorbent then for all $v \in \mathcal{H}$ there exists $a \in \mathbb{R}$ such that $v \in [(\varepsilon^b)_\varepsilon]A^\circ = ([(\varepsilon^{-b})_\varepsilon]A)^\circ$ for all $b \le a$. As a consequence, recalling that a typical neighborhood of the origin in $(\mathcal{G}, \sigma(\mathcal{G},\mathcal{H}))$ is of the form $U := \{u \in \mathcal{G} :\ \max_{i=1,\dots,n} |\mathsf{b}(u, v_i)|_\mathrm{e} \le \eta\} = \{u \in \mathcal{G} :\ \max_{i=1,\dots,n} |\mathsf{b}([(\varepsilon^{\log\eta})_\varepsilon]u, v_i)|_\mathrm{e} \le 1\}$ we find a in $\mathbb{R}$ such that $[(\varepsilon^{-b})_\varepsilon]A \subseteq U$ provided $b \le a + \log\eta$. This inclusion shows that A is bounded for $\sigma(\mathcal{G},\mathcal{H})$. Conversely if A is bounded then for all $v \in \mathcal{H}$ there exists $a \in \mathbb{R}$ such that A is contained in $[(\varepsilon^b)_\varepsilon]\{u \in \mathcal{G} :\ |\mathsf{b}(u, v)|_\mathrm{e} \le 1\}$ for all $b \le a$. By the first statement of this proposition we conclude that A° absorbs every v in $\mathcal{H}$ since

$$
\begin{aligned}
[(\varepsilon^{-b})_\varepsilon]v \in [(\varepsilon^{-b})_\varepsilon]\{u \in \mathcal{G} :\ |\mathsf{b}(u, v)|_\mathrm{e} \le 1\}^\circ \\
= ([(\varepsilon^{b})_\varepsilon]\{u \in \mathcal{G} :\ |\mathsf{b}(u, v)|_\mathrm{e} \le 1\})^\circ \subseteq A^\circ
\end{aligned}
$$

for any b smaller than a. □

By Proposition 2.4 the polar set of a $\sigma(\mathcal{G},\mathcal{H})$-bounded subset of $\mathcal{G}$ is absorbent and absolutely convex. Hence its gauge defines an ultra-pseudo-seminorm on $\mathcal{H}$.

Definition 2.5. *Let $(\mathcal{G},\mathcal{H},\mathsf{b})$ be a pairing of $\widetilde{\mathbb{C}}$-modules. A topology on $\mathcal{H}$ is said to be* polar *if it is determined by the family of ultra-pseudo-seminorms $\{\mathcal{P}_{A^\circ}\}_{A\in\mathcal{A}}$ where $\mathcal{A}$ is a collection of $\sigma(\mathcal{G},\mathcal{H})$-bounded subsets of $\mathcal{G}$. When $\mathcal{A}$ is the collection of all $\sigma(\mathcal{G},\mathcal{H})$-bounded subsets of $\mathcal{G}$ then the corresponding polar topology is called* strong topology *and denoted by $\beta(\mathcal{H},\mathcal{G})$.*

Note that $[(\varepsilon^{-b})_\varepsilon]v \in A^\circ$ if and only if $\sup_{u\in A} |\mathsf{b}(u, v)|_\mathrm{e} \le \mathrm{e}^{-b}$. It follows that $\mathcal{P}_{A^\circ}(v) = \sup_{u\in A} |\mathsf{b}(u, v)|_\mathrm{e}$ for every $\sigma(\mathcal{G},\mathcal{H})$-bounded subset A of $\mathcal{G}$. It is clear that the strong topology $\beta(\mathcal{H},\mathcal{G})$ is finer than the weak topology $\sigma(\mathcal{H},\mathcal{G})$.

We now deal with a particular type of locally convex topological $\widetilde{\mathbb{C}}$-modules whose topological duals have some interesting properties as we shall see in Proposition 2.10.

Definition 2.6. *In a topological $\widetilde{\mathbb{C}}$-module $\mathcal{G}$ a set S is said to be* bornivorous *if it absorbs every bounded subset of $\mathcal{G}$, that is, for all bounded subsets A of $\mathcal{G}$ there exists $a \in \mathbb{R}$ such that $A \subseteq [(\varepsilon^b)_\varepsilon]S$ for every $b \le a$.*

Definition 2.7. *A locally convex topological $\widetilde{\mathbb{C}}$-module $\mathcal{G}$ is* bornological *if every balanced, convex and bornivorous subset of $\mathcal{G}$ is a neighborhood of the origin.*

In the sequel we discuss the main result on bornological $\widetilde{\mathbb{C}}$-modules concerning bounded $\widetilde{\mathbb{C}}$-linear maps and we give some examples.

Proposition 2.8. *Let $\mathcal{G}$ be a bornological locally convex topological $\widetilde{\mathbb{C}}$-module and $\mathcal{H}$ be an arbitrary locally convex topological $\widetilde{\mathbb{C}}$-module. If T is a $\widetilde{\mathbb{C}}$-linear bounded map from $\mathcal{G}$ into $\mathcal{H}$ then T is continuous.*

Proof. Let V be an absolutely convex neighborhood of 0 in $\mathcal{H}$ and A be a bounded subset of $\mathcal{G}$. By hypothesis $T(A)$ is bounded in $\mathcal{H}$, i.e. there exists $a \in \mathbb{R}$ such that $T(A) \subseteq [(\varepsilon^b)_\varepsilon]V$ for all $b \le a$. $T^{-1}(V)$ is absolutely convex in

$\mathcal{G}$ and, as proven above, it absorbs every bounded subset of $\mathcal{G}$. Therefore, since $\mathcal{G}$ is bornological, $T^{-1}(V)$ is a neighborhood of 0 in $\mathcal{G}$ and T is continuous. $\square$

Proposition 2.9.

(i) Every locally convex topological $\widetilde{\mathbb{C}}$-module $\mathcal{G}$ which has a countable base of neighborhoods of the origin is bornological.

(ii) The inductive limit $\mathcal{G}$ of a family of bornological locally convex topological $\widetilde{\mathbb{C}}$-modules $(\mathcal{G}_\gamma)_{\gamma\in\Gamma}$ is bornological.

Proof. We easily adapt the proof of the corresponding results for locally convex topological vector spaces presented in [28, Chapter 3, Section 7, Propositions 3,4].
(i) Let $\mathcal{G}$ be a locally convex topological $\widetilde{\mathbb{C}}$-module with a countable base of neighborhoods of the origin. We may choose a balanced base of neighborhoods of the origin $(V_n)_n$ such that $V_{n+1} \subseteq V_n$ for all $n \in \mathbb{N}$. Let U be a balanced, convex and bornivorous subset of $\mathcal{G}$. We want to prove that U contains some $[(\varepsilon^n)_\varepsilon]V_n$. Assume that U does not contain any $[(\varepsilon^n)_\varepsilon]V_n$. This means that we find a sequence $(u_n)_n$ of points $u_n \in ([(\varepsilon^n)_\varepsilon]V_n) \cap (\mathcal{G} \setminus U)$. Now by construction $[(\varepsilon^{-n})_\varepsilon]u_n$ converges to 0 in $\mathcal{G}$ and so the set $A := \{[(\varepsilon^{-n})_\varepsilon]u_n,\ n \in \mathbb{N}\}$ is bounded. But U does not absorb A because if there existed $a \in \mathbb{R}$ such that $A \subseteq [(\varepsilon^a)_\varepsilon]U$ then $u_n \in [(\varepsilon^{n+a})_\varepsilon]U \subseteq U$ for n large enough, in contradiction to our choice of the sequence $(u_n)_n$. Thus U is a neighborhood of 0 in $\mathcal{G}$.
(ii) Let $\iota_\gamma : \mathcal{G}_\gamma \to \mathcal{G}$ be the family of $\widetilde{\mathbb{C}}$-linear maps which defines the inductive limit topology on $\mathcal{G}$ and U be an absolutely convex and bornivorous subset of $\mathcal{G}$. Hence $\iota_\gamma^{-1}(U)$ is absolutely convex in $\mathcal{G}_\gamma$ and by continuity of ι_γ, if A_γ is bounded in $\mathcal{G}_\gamma$ then $\iota_\gamma(A_\gamma)$ is bounded in $\mathcal{G}$. By Definition 2.6 there exists $a_\gamma \in \mathbb{R}$ such that $\iota_\gamma(A_\gamma) \subseteq [(\varepsilon^b)_\varepsilon]U$ for all $b \le a_\gamma$ that is $A_\gamma \subseteq [(\varepsilon^b)_\varepsilon]\iota_\gamma^{-1}(U)$. We have proved that $\iota_\gamma^{-1}(U)$ is balanced, convex and bornivorous and since $\mathcal{G}_\gamma$ is bornological, $\iota_\gamma^{-1}(U)$ is a neighborhood of 0 in $\mathcal{G}_\gamma$. This tells us that U is a neighborhood of 0 in $\mathcal{G}$. $\square$

We conclude this section by considering the pairing formed by a topological $\widetilde{\mathbb{C}}$-module $\mathcal{G}$ and its topological dual $\mathcal{L}(\mathcal{G},\widetilde{\mathbb{C}})$. We know that $\mathcal{L}(\mathcal{G},\widetilde{\mathbb{C}})$ can be endowed with the separated topologies $\sigma(\mathcal{L}(\mathcal{G},\widetilde{\mathbb{C}}),\mathcal{G})$ and $\beta(\mathcal{L}(\mathcal{G},\widetilde{\mathbb{C}}),\mathcal{G})$. Since every ultra-pseudo-seminorm defining $\sigma(\mathcal{G},\mathcal{L}(\mathcal{G},\widetilde{\mathbb{C}}))$ is continuous for the original topology τ on $\mathcal{G}$, $\sigma(\mathcal{G},\mathcal{L}(\mathcal{G},\widetilde{\mathbb{C}})$ is coarser than τ. In some particular cases the strong topology turns $\mathcal{L}(\mathcal{G},\widetilde{\mathbb{C}})$ into a complete topological $\widetilde{\mathbb{C}}$-module.

Proposition 2.10. *Let $\mathcal{G}$ be a bornological locally convex topological $\widetilde{\mathbb{C}}$-module. Then $\mathcal{L}(\mathcal{G},\widetilde{\mathbb{C}})$ with the topology $\beta(\mathcal{L}(\mathcal{G},\widetilde{\mathbb{C}}),\mathcal{G})$ is complete.*

Proof. We shall show that every Cauchy filter $\mathcal{F}$ on $\mathcal{L}(\mathcal{G},\widetilde{\mathbb{C}})$ is convergent. First of all by the same reasoning as in Proposition 2.2, for all $u \in \mathcal{G}$ the filter $\mathcal{F}_u$ generated by $\{X_u\}_{X\in\mathcal{F}}$ where $X_u := \{Tu : T \in X\}$, is a Cauchy filter on $\widetilde{\mathbb{C}}$ and it converges to some $F(u) \in \widetilde{\mathbb{C}}$. The function $F : \mathcal{G} \to \widetilde{\mathbb{C}} : u \to F(u)$ is $\widetilde{\mathbb{C}}$-linear. Moreover F is continuous on $\mathcal{G}$. In fact every bounded subset A of $\mathcal{G}$ is $\sigma(\mathcal{G},\mathcal{L}(\mathcal{G},\widetilde{\mathbb{C}}))$-bounded and by definition of a Cauchy filter and the strong

topology on $\mathrm{L}(\mathcal{G},\widetilde{\mathbb{C}})$, there exists $X \in \mathcal{F}$ such that $X - X \subseteq A^\circ$. This means that for all $T, T' \in X$ and for all $u \in A$ we have $|T(u) - T'(u)|_\mathrm{e} \le 1$. In particular we find a constant $c > 0$ such that $|T(u)|_\mathrm{e} \le c$ for all T in X and for all $u \in A$. On the other hand given $u \in A$, $F(u)$ adheres to $\mathcal{F}_u$ and therefore there exists $T' \in X$ such that $|F(u) - T'(u)|_\mathrm{e} \le c$. Thus for all $u \in A$ we may write

$$|F(u)|_\mathrm{e} \le \max\{|F(u) - T'(u)|_\mathrm{e}, |T'(u)|_\mathrm{e}\} \le c$$

which yields that $F(A)$ is a bounded subset of $\widetilde{\mathbb{C}}$. Since $\mathcal{G}$ is bornological, the bounded $\widetilde{\mathbb{C}}$-linear map F is continuous.

We complete the proof by proving that $\mathcal{F}$ converges to F in the strong topology $\beta(\mathrm{L}(\mathcal{G},\widetilde{\mathbb{C}}),\mathcal{G})$. For all $\sigma(\mathcal{G},\mathrm{L}(\mathcal{G},\widetilde{\mathbb{C}}))$-bounded subsets A of $\mathcal{G}$ and for all $\eta > 0$ there exists $X \in \mathcal{F}$ such that $|T(u) - T'(u)|_\mathrm{e} < \eta$ for all u in A and T, T' in X. Since $\mathcal{F}_u \to F(u)$, for all $u \in A$ there exists $T' \in X$ such that $|F(u) - T'(u)|_\mathrm{e} < \eta$. Then for all $T \in X$ and $u \in A$

$$|F(u) - T(u)|_\mathrm{e} \le \max\{|F(u) - T'(u)|_\mathrm{e}, |T'(u) - T(u)|_\mathrm{e}\} < \eta$$

or in other words $\mathcal{P}_{A^\circ}(F - T) < \eta$. This implies the inclusion $X \subseteq F + [(\varepsilon^{-\log\eta})_\varepsilon]A^\circ$. A typical neighborhood of the origin in $\beta(\mathrm{L}(\mathcal{G},\widetilde{\mathbb{C}}),\mathcal{G})$ is given by $\{T:\ \max_{i=1,\dots,N}\mathcal{P}_{A_i^\circ}(T) < \eta\}$. Hence, from the previous considerations there exists $X = \cap_{i=1,\dots N} X_i$, $X_i \in \mathcal{F}$, such that $X \subseteq F + [(\varepsilon^{-\log\eta})_\varepsilon] \cap_{i=1,\dots,N} A_i^\circ$, which proves our assertion. $\square$

Remark 2.11. When $\mathcal{G}$ is a topological $\widetilde{\mathbb{C}}$-module we can restrict the family of $\sigma(\mathcal{G},\mathrm{L}(\mathcal{G},\widetilde{\mathbb{C}}))$-bounded subsets which defines the strong topology $\beta(\mathrm{L}(\mathcal{G},\widetilde{\mathbb{C}}),\mathcal{G})$ to the family of bounded subsets of $\mathcal{G}$. The corresponding polar topology is called topology of uniform convergence on bounded subsets of $\mathcal{G}$ and denoted by $\beta_b(\mathrm{L}(\mathcal{G},\widetilde{\mathbb{C}}),\mathcal{G})$ here. Clearly $\beta_b(\mathrm{L}(\mathcal{G},\widetilde{\mathbb{C}}),\mathcal{G})$ is separated and coarser then $\beta(\mathrm{L}(\mathcal{G},\widetilde{\mathbb{C}}),\mathcal{G})$. A careful inspection of the proof of Proposition 2.10 shows that when $\mathcal{G}$ is a bornological locally convex topological $\widetilde{\mathbb{C}}$-module then $\mathrm{L}(\mathcal{G},\widetilde{\mathbb{C}})$ is complete for the topology $\beta_b(\mathrm{L}(\mathcal{G},\widetilde{\mathbb{C}}),\mathcal{G})$. In Section 3.3 we shall prove that the topologies $\beta_b(\mathrm{L}(\mathcal{G},\widetilde{\mathbb{C}}),\mathcal{G})$ and $\beta(\mathrm{L}(\mathcal{G},\widetilde{\mathbb{C}}),\mathcal{G})$ coincide for a certain particular class of ultra-pseudo-normed $\widetilde{\mathbb{C}}$-modules. The general issue concerning locally convex topological $\widetilde{\mathbb{C}}$-modules remains open.

We conclude our investigation into the properties of the topological dual of a locally convex topological $\widetilde{\mathbb{C}}$-module by looking at convergent sequences. More precisely we will prove that under suitable hypotheses on $\mathcal{G}$, if a sequence of $\widetilde{\mathbb{C}}$-linear continuous maps $T_n : \mathcal{G} \to \widetilde{\mathbb{C}}$ is pointwise convergent to some $T : \mathcal{G} \to \widetilde{\mathbb{C}}$ then T is itself an element of the dual $\mathrm{L}(\mathcal{G},\widetilde{\mathbb{C}})$. This requires some preliminary notions concerning barrels and barrelled $\widetilde{\mathbb{C}}$-modules.

Definition 2.12. *Let $\mathcal{G}$ be a locally convex topological $\widetilde{\mathbb{C}}$-module. An absorbent, balanced, convex and closed subset of $\mathcal{G}$ is said to be a* barrel. *A locally convex topological $\widetilde{\mathbb{C}}$-module is* barrelled *if every barrel is a neighborhood of the origin.*

We recall that a subset A of $\mathrm{L}(\mathcal{G},\widetilde{\mathbb{C}})$ ($\mathcal{G}$ topological $\widetilde{\mathbb{C}}$-module) is *equicontinuous at $u_0 \in \mathcal{G}$* if for every W neighborhood of the origin in $\widetilde{\mathbb{C}}$ there exists a neighborhood U of u_0 in $\mathcal{G}$ such that $T(u) - T(u_0) \in W$ for all $u \in U$ and $T \in A$. A is equicontinuous if it is equicontinuous at every point of $\mathcal{G}$.

There exists a relationship among barrels of $\mathcal{G}$, $\sigma(\mathcal{L}(\mathcal{G},\widetilde{\mathbb{C}}),\mathcal{G})$-bounded subsets and equicontinuous subsets of $\mathcal{L}(\mathcal{G},\widetilde{\mathbb{C}})$.

Proposition 2.13.

(i) Let $\mathcal{G}$ be a locally convex topological $\widetilde{\mathbb{C}}$-module. If the subset $A \subseteq \mathcal{L}(\mathcal{G},\widetilde{\mathbb{C}})$ is $\sigma(\mathcal{L}(\mathcal{G},\widetilde{\mathbb{C}}),\mathcal{G})$-bounded then there exists a barrel B in $\mathcal{G}$ such that $A \subseteq B^\circ$.

(ii) If $\mathcal{G}$ is a barrelled locally convex topological $\widetilde{\mathbb{C}}$-module then every $A \subseteq \mathcal{L}(\mathcal{G},\widetilde{\mathbb{C}})$ which is bounded for $\sigma(\mathcal{L}(\mathcal{G},\widetilde{\mathbb{C}}),\mathcal{G})$ is equicontinuous.

Proof. (i) If A is $\sigma(\mathcal{L}(\mathcal{G},\widetilde{\mathbb{C}}),\mathcal{G})$-bounded then by Proposition 2.4 its polar A° is absorbent, balanced and convex. Moreover, $A^\circ = \cap_{T\in A} A_T$ where $A_T := \{u \in \mathcal{G} : |T(u)|_{\mathrm{e}} \le 1\}$ is closed in $\mathcal{G}$ by continuity of T. Hence $B := A^\circ$ is a barrel of $\mathcal{G}$ such that $A \subseteq B^\circ$.
(ii) By assertion (i) every $\sigma(\mathcal{L}(\mathcal{G},\widetilde{\mathbb{C}}),\mathcal{G})$-bounded subset A is contained in some B° where B is a barrel of $\mathcal{G}$. This means that $|T([(\varepsilon^{-\log\eta})_\varepsilon]u)|_{\mathrm{e}} \le \eta$ for all $T \in A$, $u \in B$ and $\eta > 0$. Since $\mathcal{G}$ is barrelled B is a neighborhood of 0 and by the estimate above A is an equicontinuous subset of $\mathcal{L}(\mathcal{G},\widetilde{\mathbb{C}})$. □

We now give some examples of barrelled locally convex topological $\widetilde{\mathbb{C}}$-modules. We recall that if a *Baire space* is the union of a countable family of closed subsets S_n at least one set S_n has nonempty interior. Baire's Theorem says that a complete metrizable topological space is a Baire space. Hence every Fréchet $\widetilde{\mathbb{C}}$-module is a Baire space.

Proposition 2.14. *A locally convex topological $\widetilde{\mathbb{C}}$-module $\mathcal{G}$ which is a Baire space is barrelled.*

Proof. Let B be an absorbent, balanced, convex and closed subset of $\mathcal{G}$. Since it is absorbent we may write $\mathcal{G} = \cup_{n\in\mathbb{N}}[(\varepsilon^{-n})_\varepsilon]B$, where each $[(\varepsilon^{-n})_\varepsilon]B$ is closed in $\mathcal{G}$. $\mathcal{G}$ is a Baire space. Hence there exists some $[(\varepsilon^{-n})_\varepsilon]B$ with nonempty interior and from the continuity of the scalar multiplication $\mathcal{G} \to \mathcal{G} : u \to [(\varepsilon^{-n})_\varepsilon]u$ we conclude that $\mathrm{int}(B) \neq \emptyset$. Let $u_0 \in \mathrm{int}(B)$. We find a neighborhood V of 0 such that $u_0 + V \subseteq B$ and since B is balanced $-u_0$ belongs to B. Hence by the convexity of B, $V \subseteq u_0 + V - u_0 \subseteq B + B \subseteq B$ which yields that B is a neighborhood of 0 in $\mathcal{G}$. □

Proposition 2.14 allows us to say that every Fréchet $\widetilde{\mathbb{C}}$-module is a barrelled locally convex topological $\widetilde{\mathbb{C}}$-module. The same conclusion holds when we consider an inductive limit procedure.

Proposition 2.15. *The inductive limit $\mathcal{G}$ of a family of barrelled locally convex topological $\widetilde{\mathbb{C}}$-modules $(\mathcal{G}_\gamma)_{\gamma\in\Gamma}$ is barrelled.*

The proof of Proposition 2.15 is left to the reader since it is an elementary application of the definition of barrelled locally convex topological $\widetilde{\mathbb{C}}$-module and the continuity properties. Before dealing with sequences $(T_n)_n$ in the dual of a barrelled locally convex topological $\widetilde{\mathbb{C}}$-module which are pointwise convergent to some map $T : \mathcal{G} \to \widetilde{\mathbb{C}}$ we state a preparatory lemma.

Lemma 2.16. *Let $\mathcal{G}$ be a topological $\widetilde{\mathbb{C}}$-module, M an equicontinuous subset of $\mathcal{L}(\mathcal{G},\widetilde{\mathbb{C}})$ and $\mathcal{F}$ a filter on M. Assume that for all $u \in \mathcal{G}$, the filter $\mathcal{F}_u$ converges to some $F(u) \in \widetilde{\mathbb{C}}$. Then the map $F : u \to F(u)$ belongs to $\mathcal{L}(\mathcal{G},\widetilde{\mathbb{C}})$.*

Proof. As already observed in the proof of Proposition 2.2, $\mathcal{F}_u$, the filter generated by $\{X_u\}_{X\in\mathcal{F}}$, $X_u := \{T(u), T \in X\}$, which is convergent to $F(u)$, provides a $\widetilde{\mathbb{C}}$-linear map $F : \mathcal{G} \to \widetilde{\mathbb{C}}$. Since M is equicontinuous we have that for all $\eta > 0$ there exists a neighborhood U of the origin in $\mathcal{G}$ such that $|T(u)|_{\mathrm{e}} \le \eta$ for all $u \in U$ and $T \in M$. By $\mathcal{F}_u \to F(u)$ it follows that for all $u \in U$ and $\eta > 0$ there exists $T' \in M$ such that $|T'(u) - F(u)|_{\mathrm{e}} \le \eta$. The estimate $|F(u)|_{\mathrm{e}} \le \max\{|T'(u) - F(u)|_{\mathrm{e}}, |T'(u)|_{\mathrm{e}}\} \le \eta$ valid on U entails that F is continuous at 0 and therefore continuous on $\mathcal{G}$. □

Proposition 2.17. *Let $\mathcal{G}$ be a barrelled locally convex topological $\widetilde{\mathbb{C}}$-module. Let $\mathcal{F}$ be a filter on $\mathcal{L}(\mathcal{G},\widetilde{\mathbb{C}})$ which contains a $\sigma(\mathcal{L}(\mathcal{G},\widetilde{\mathbb{C}}),\mathcal{G})$-bounded subset of $\mathcal{L}(\mathcal{G},\widetilde{\mathbb{C}})$. Assume that the filter $\mathcal{F}_u$ converges to some $F(u) \in \widetilde{\mathbb{C}}$. Then the map $F : u \to F(u)$ belongs to $\mathcal{L}(\mathcal{G},\widetilde{\mathbb{C}})$.*

Proof. By assertion (ii) in Proposition 2.13 we know that $\mathcal{F}$ contains an equicontinuous subset M of $\mathcal{L}(\mathcal{G},\widetilde{\mathbb{C}})$. Let $\mathcal{O} := \{Y \subseteq M : \exists X \in \mathcal{F}\ X \cap M \subseteq Y\}$ be the filter induced by $\mathcal{F}$ on M. $\mathcal{O}_u$ converges to $F(u)$ for all $u \in \mathcal{G}$. An application of Lemma 2.16 proves that F is a $\widetilde{\mathbb{C}}$-linear and continuous map on $\mathcal{G}$. □

Corollary 2.18. *Let $\mathcal{G}$ be a barrelled locally convex topological $\widetilde{\mathbb{C}}$-module. Suppose that $(T_n)_n$ is a sequence in $\mathcal{L}(\mathcal{G},\widetilde{\mathbb{C}})$ such that for every $u \in \mathcal{G}$ the sequence $(T_n(u))_n$ converges to some $T(u) \in \widetilde{\mathbb{C}}$. Then $T : \mathcal{G} \to \widetilde{\mathbb{C}} : u \to T(u)$ belongs to $\mathcal{L}(\mathcal{G},\widetilde{\mathbb{C}})$.*

Proof. The elementary filter associated with the sequence $(T_n)_n$ i.e. the filter $\mathcal{F}$ generated by $X_N := \{T_n,\ n \ge N\}$, $N \in \mathbb{N}$, contains a $\sigma(\mathcal{L}(\mathcal{G},\widetilde{\mathbb{C}}),\mathcal{G})$-bounded subset of $\mathcal{L}(\mathcal{G},\widetilde{\mathbb{C}})$. In fact by $T_n(u) \to T(u)$ each X_N is $\sigma(\mathcal{L}(\mathcal{G},\widetilde{\mathbb{C}}),\mathcal{G})$-bounded and by construction $\mathcal{F}_u \to T(u)$ for all $u \in \mathcal{G}$. By Proposition 2.17 we conclude that $T \in \mathcal{L}(\mathcal{G},\widetilde{\mathbb{C}})$. □

Corollary 2.19. *If $\mathcal{G}$ is a barrelled locally convex topological $\widetilde{\mathbb{C}}$-module then the topological dual $\mathcal{L}(\mathcal{G},\widetilde{\mathbb{C}})$ endowed with the weak-topology $\sigma(\mathcal{L}(\mathcal{G},\widetilde{\mathbb{C}}),\mathcal{G})$ is quasi-complete.*

Proof. We have to show that every closed and bounded subset M of $\mathcal{L}(\mathcal{G},\widetilde{\mathbb{C}})$ is complete for the topology $\sigma(\mathcal{L}(\mathcal{G},\widetilde{\mathbb{C}}),\mathcal{G})$. Let $\mathcal{F}$ be a Cauchy filter on M. $\mathcal{F}$ generates a Cauchy filter $\mathcal{F}'$ on $\mathcal{L}(\mathcal{G},\widetilde{\mathbb{C}})$ which contains a $\sigma(\mathcal{L}(\mathcal{G},\widetilde{\mathbb{C}}),\mathcal{G})$-bounded subset of $\mathcal{L}(\mathcal{G},\widetilde{\mathbb{C}})$ and a Cauchy filter $\mathcal{F}''$ on $(L(\mathcal{G},\widetilde{\mathbb{C}}),\sigma(L(\mathcal{G},\widetilde{\mathbb{C}}),\mathcal{G}))$. By Proposition 2.2 there exists $F \in L(\mathcal{G},\widetilde{\mathbb{C}})$ such that $\mathcal{F}'' \to F$ in $L(\mathcal{G},\widetilde{\mathbb{C}})$ and consequently $\mathcal{F}'_u \to F(u)$ for all $u \in \mathcal{G}$. At this point Proposition 2.17 allows to conclude that $F \in \mathcal{L}(\mathcal{G},\widetilde{\mathbb{C}})$ and $\mathcal{F}' \to F$ in $\mathcal{L}(\mathcal{G},\widetilde{\mathbb{C}})$. Since $\mathcal{F}$ is a filter on M and M is $\sigma(\mathcal{L}(\mathcal{G},\widetilde{\mathbb{C}}),\mathcal{G})$-closed, F itself belongs to M and $\mathcal{F} \to F$. □

3 $\widetilde{\mathbb{C}}$-modules of generalized functions based on a locally convex topological vector space

In this part of the paper we focus our attention on a relevant class of examples of $\widetilde{\mathbb{C}}$-modules, whose general theory was developed in the previous sections. In the literature there already exist papers on topologies, generalized functions and applications cf. [1, 4, 35] which consider spaces of generalized functions $\mathcal{G}_E$ based on a locally convex topological vector space E and define topologies in terms of valuations and ultra-pseudo-seminorms [32, 46, 47, 48, 50]. The topological background provided by Sections 1 and 2 allows us to consider $\mathcal{G}_E$ as an element of the larger family of locally convex topological $\widetilde{\mathbb{C}}$-modules and to deal with issues as boundedness, completeness and topological duals. For the sake of exposition we organize the following notions and results in three subsections.

3.1 Definition and basic properties of $\mathcal{G}_E$

Definition 3.1. *Let E be a locally convex topological vector space topologized through the family of seminorms $\{p_i\}_{i\in I}$. The elements of*

$$\mathcal{M}_E := \{(u_\varepsilon)_\varepsilon \in E^{(0,1]} : \forall i \in I\ \exists N \in \mathbb{N} \quad p_i(u_\varepsilon) = O(\varepsilon^{-N})\ as\, \varepsilon \to 0\} \tag{3.10}$$

and

$$\mathcal{N}_E := \{(u_\varepsilon)_\varepsilon \in E^{(0,1]} : \forall i \in I\ \forall q \in \mathbb{N} \quad p_i(u_\varepsilon) = O(\varepsilon^{q})\ as\, \varepsilon \to 0\}, \tag{3.11}$$

are called E-moderate and E-negligible, respectively. We define the space of generalized functions based on E *as the factor space $\mathcal{G}_E := \mathcal{M}_E/\mathcal{N}_E$.*

It is clear that the definition of $\mathcal{G}_E$ does not depend on the family of seminorms which determines the locally convex topology of E. We adopt the notation $u = [(u_\varepsilon)_\varepsilon]$ for the class u of $(u_\varepsilon)_\varepsilon$ in $\mathcal{G}_E$ and we embed E into $\mathcal{G}_E$ via the constant embedding $f \to [(f)_\varepsilon]$. By the properties of seminorms on E we may define the product between complex generalized numbers and elements of $\mathcal{G}_E$ via the map $\widetilde{\mathbb{C}} \times \mathcal{G}_E \to \mathcal{G}_E : ([(\lambda_\varepsilon)_\varepsilon], [(u_\varepsilon)_\varepsilon]) \to [(\lambda_\varepsilon u_\varepsilon)_\varepsilon]$, which equips $\mathcal{G}_E$ with the structure of a $\widetilde{\mathbb{C}}$-module.

Since the growth in ε of an E-moderate net is estimated in terms of any seminorm p_i of E, it is natural to introduce the *p_i-valuation* of $(u_\varepsilon)_\varepsilon \in \mathcal{M}_E$ as

$$\mathrm{v}_{p_i}((u_\varepsilon)_\varepsilon) := \sup\{b \in \mathbb{R} : \quad p_i(u_\varepsilon) = O(\varepsilon^b) \quad \text{as}\, \varepsilon \to 0\}. \tag{3.12}$$

Note that $\mathrm{v}_{p_i}((u_\varepsilon)_\varepsilon) = \mathrm{v}((p_i(u_\varepsilon))_\varepsilon)$ where the function v in (1.1) gives the valuation on $\widetilde{\mathbb{C}}$. Clearly v_{p_i} maps $\mathcal{M}_E$ into $(-\infty, +\infty]$ and the following properties hold:

(i) $\mathrm{v}_{p_i}((u_\varepsilon)_\varepsilon) = +\infty$ for all $i \in I$ if and only if $(u_\varepsilon)_\varepsilon \in \mathcal{N}_E$,

(ii) $\mathrm{v}_{p_i}((\lambda_\varepsilon u_\varepsilon)_\varepsilon) \geq \mathrm{v}((\lambda_\varepsilon)_\varepsilon) + \mathrm{v}_{p_i}((u_\varepsilon)_\varepsilon)$ for all $(\lambda_\varepsilon)_\varepsilon \in \mathcal{E}_M$ and $(u_\varepsilon)_\varepsilon \in \mathcal{M}_E$,

(ii)' $\mathrm{v}_{p_i}((\lambda_\varepsilon u_\varepsilon)_\varepsilon) = \mathrm{v}((\lambda_\varepsilon)_\varepsilon) + \mathrm{v}_{p_i}((u_\varepsilon)_\varepsilon)$ for all $(\lambda_\varepsilon)_\varepsilon = (c\varepsilon^b)_\varepsilon$, $c \in \mathbb{C}$, $b \in \mathbb{R}$,

(iii) $\mathrm{v}_{p_i}((u_\varepsilon)_\varepsilon + (v_\varepsilon)_\varepsilon) \geq \min\{\mathrm{v}_{p_i}((u_\varepsilon)_\varepsilon), \mathrm{v}_{p_i}((v_\varepsilon)_\varepsilon)\}$.

Assertion (i) combined with (iii) shows that $\mathrm{v}_{p_i}((u_\varepsilon)_\varepsilon) = \mathrm{v}_{p_i}((u'_\varepsilon)_\varepsilon)$ if $(u_\varepsilon - u'_\varepsilon)_\varepsilon$ is E-negligible. This means that we can use (3.12) for defining the *p_i-valuation* $\mathrm{v}_{p_i}(u) = \mathrm{v}_{p_i}((u_\varepsilon)_\varepsilon)$ of a generalized function $u = [(u_\varepsilon)_\varepsilon] \in \mathcal{G}_E$.

v_{p_i} is a valuation in the sense of Definition 1.8 and thus $\mathcal{P}_i(u) := \mathrm{e}^{-\mathrm{v}_{p_i}(u)}$ is an ultra-pseudo-seminorm on the $\widetilde{\mathbb{C}}$-module $\mathcal{G}_E$. By Theorem 1.10, $\mathcal{G}_E$ endowed with the topology of the ultra-pseudo-seminorms $\{\mathcal{P}_i\}_{i\in I}$ is a locally convex topological $\widetilde{\mathbb{C}}$-module. Following [32, 46, 47, 48] we use the adjective "sharp" for the topology induced by the ultra-pseudo-seminorms $\{\mathcal{P}_i\}_{i\in I}$. The sharp topology on $\mathcal{G}_E$, here denoted by $\tau_\sharp$ is independent of the choice of the family of seminorms which determines the original locally convex topology on E. The structure of the subspace $\mathcal{N}_E$ has some interesting influence on $\tau_\sharp$.

Proposition 3.2. *$(\mathcal{G}_E, \tau_\sharp)$ is a separated locally convex topological $\widetilde{\mathbb{C}}$-module.*

Proof. By definition of $\mathcal{N}_E$ if $u \neq 0$ in $\mathcal{G}_E$ then $\mathrm{v}_{p_i}((u_\varepsilon)_\varepsilon) \neq +\infty$ for some $i \in I$. This means that $\mathcal{P}_i(u) > 0$ and by Proposition 1.11 $\tau_\sharp$ is a separated topology. □

Proposition 3.3. *Let E be a locally convex topological vector space.*

(i) If E is topologized through an increasing sequence $\{p_i\}_{i\in\mathbb{N}}$ of seminorms and

$$\mathcal{N}_E = \{(u_\varepsilon)_\varepsilon \in \mathcal{M}_E : \forall q \in \mathbb{N} \quad p_0(u_\varepsilon) = O(\varepsilon^q) \ as\, \varepsilon \to 0\}, \tag{3.13}$$

then each $\mathcal{P}_i$ is an ultra-pseudo-norm on $\mathcal{G}_E$.

(ii) If E has a countable base of neighborhoods of the origin then $\mathcal{G}_E$ with the sharp topology is metrizable.

Proof. Concerning the first assertion we have to prove that $\mathcal{P}_i(u) = 0$ implies $u = 0$ in $\mathcal{G}_E$. From $\mathcal{P}_i(u) = 0$ it follows that $p_i(u_\varepsilon) = O(\varepsilon^q)$ for all $q \in \mathbb{N}$. Since $p_0(u_\varepsilon) \leq p_i(u_\varepsilon)$, (3.13) leads to $(u_\varepsilon)_\varepsilon \in \mathcal{N}_E$. Combining Proposition 3.2 with the assumption (ii), we obtain that $\mathcal{G}_E$ is a separated locally convex topological $\widetilde{\mathbb{C}}$-module with a countable base of neighborhoods of the origin. Hence by Theorem 1.14 it is metrizable. □

Proposition 3.4. *If E is a locally convex topological vector space with a countable base of neighborhoods of the origin then $\mathcal{G}_E$ with the sharp topology $\tau_\sharp$ is complete.*

This result, in terms of convergence of Cauchy sequences, was already proven in [47, Proposition 2.1]. For the convenience of the reader we add some details to the sketch of the proof given there.

Proof. As shown in Proposition 3.3 $\mathcal{G}_E$ is metrizable and therefore by Remark 1.30 it is sufficient to prove that any Cauchy sequence in $\mathcal{G}_E$ is convergent. It is not restrictive to assume that E is topologized through an increasing sequence of seminorms $\{p_k\}_{k\in\mathbb{N}}$. If $(u_n)_n$ is a Cauchy sequence in $\mathcal{G}_E$ we may extract a subsequence $(u_{n_k})_k$ such that $\mathrm{v}_{p_k}((u_{n_{k+1},\varepsilon} - u_{n_k,\varepsilon})_\varepsilon) > k$ for all $k \in \mathbb{N}$. As in the proof of Proposition 1.31 we obtain a decreasing sequence $\varepsilon_k \searrow 0$, $\varepsilon_k \le 2^{-k}$ such that $p_k(u_{n_{k+1},\varepsilon} - u_{n_k,\varepsilon}) \le \varepsilon^k$ for all $\varepsilon \in (0, \varepsilon_k)$. Let

$$h_{k,\varepsilon} = \begin{cases} u_{n_{k+1},\varepsilon} - u_{n_k,\varepsilon} & \varepsilon \in (0,\varepsilon_k), \\ 0 & \varepsilon \in [\varepsilon_k, 1]. \end{cases}$$

Obviously $(h_{k,\varepsilon})_\varepsilon$ belongs to $\mathcal{M}_E$ and for all $k' \le k$, $p_{k'}(h_{k,\varepsilon}) \le \varepsilon^k$ on the interval $(0,1]$. The sum $u_\varepsilon := u_{n_0,\varepsilon} + \sum_{k=0}^\infty h_{k,\varepsilon}$ is locally finite and E-moderate since for all $\overline{k} \in \mathbb{N}$

$$\begin{aligned} p_{\overline{k}}(u_\varepsilon) \le p_{\overline{k}}(u_{n_0,\varepsilon}) + \sum_{k=0}^{\overline{k}} p_{\overline{k}}(h_{k,\varepsilon}) + \sum_{k=\overline{k}+1}^{\infty} p_{\overline{k}}(h_{k,\varepsilon}) \\ \le p_{\overline{k}}(u_{n_0,\varepsilon}) + \sum_{k=0}^{\overline{k}} p_{\overline{k}}(h_{k,\varepsilon}) + \sum_{k=\overline{k}+1}^{\infty} \frac{1}{2^k}. \end{aligned}$$

Finally for all $\overline{k} \ge 1$ and for all $\varepsilon \in (0, \varepsilon_{\overline{k}-1})$

$$p_{\overline{k}}(u_{n_{\overline{k}},\varepsilon} - u_\varepsilon) = p_{\overline{k}}\big(-\sum_{k=\overline{k}}^{\infty} h_{k,\varepsilon}\big) \le \sum_{k=\overline{k}}^{\infty} \varepsilon^{\overline{k}-1}\varepsilon_k \le \varepsilon^{\overline{k}-1} \sum_{k=\overline{k}}^{\infty} \frac{1}{2^k}. \tag{3.14}$$

By (3.14) we conclude that for all $\overline{k} \ge 1$, for all $q \in \mathbb{N}$, for all $k \ge \max\{\overline{k}, q+1\}$ there exists $\eta \in (0,1]$ such that $p_{\overline{k}}(u_{n_k,\varepsilon} - u_\varepsilon) \le \varepsilon^q$ on $(0,\eta]$. In other words $(u_{n_k})_k$ is convergent to u in $\mathcal{G}_E$. Consequently $(u_n)_n$ itself converges to u in $\mathcal{G}_E$. $\square$

Remark 3.5. We recall that a $\widetilde{\mathbb{C}}$-module $\mathcal{G}$ is a $\widetilde{\mathbb{C}}$-*algebra* if there is given a multiplication $\mathcal{G} \times \mathcal{G} \to \mathcal{G} : (u,v) \to uv$ such that $(uv)w = u(vw)$, $u(v+w) = uv + uw$, $(u+v)w = uw + vw$, $\lambda(uv) = (\lambda u)v = u(\lambda v)$ for all u, v, w in $\mathcal{G}$ and $\lambda \in \widetilde{\mathbb{C}}$. In analogy with the theory of topological algebras we say that a $\widetilde{\mathbb{C}}$-algebra $\mathcal{G}$ is a topological $\widetilde{\mathbb{C}}$-algebra if it is equipped with a $\widetilde{\mathbb{C}}$-linear topology which makes the multiplication on $\mathcal{G}$ continuous. As an explanatory example let us consider an algebra E on $\mathbb{C}$ and a family of seminorms $\{p_i\}_{i\in I}$ on E. Assume that for all $i \in I$ there exist finite subsets I_0, I_0' of I and a constant $C_i > 0$ such that for all $u, v \in E$

$$p_i(uv) \le C_i \max_{j\in I_0} p_j(u) \ \max_{j\in I_0'} p_j(v). \tag{3.15}$$

Then $\mathcal{G}_E$ with the sharp topology determined by the ultra-pseudo-seminorms $\{\mathcal{P}_i\}_{i\in I}$ is a locally convex topological $\widetilde{\mathbb{C}}$-module and a topological $\widetilde{\mathbb{C}}$-algebra since from (3.15) it follows that

$$\mathcal{P}_i(uv) \le \max_{j\in I_0} \mathcal{P}_j(u) \ \max_{j\in I_0'} \mathcal{P}_j(v)$$

for all $i \in I$.

In the sequel we collect some examples of locally convex topological $\widetilde{\mathbb{C}}$-modules which occur in Colombeau theory. For details and explanations about Colombeau generalized functions we mainly refer to [7, 19, 32].

Example 3.6. Colombeau algebras obtained as $\widetilde{\mathbb{C}}$-modules $\mathcal{G}_E$
Particular choices of E in Definition 3.1 give us known algebras of generalized functions and the corresponding sharp topologies. This is of course the case for $E = \mathbb{C}$ and $\mathcal{G}_E = \widetilde{\mathbb{C}}$ which is an ultra-pseudo-normed $\widetilde{\mathbb{C}}$-module and more precisely a topological $\widetilde{\mathbb{C}}$-algebra.

Consider now an open subset Ω of $\mathbb{R}^n$. $E = \mathcal{E}(\Omega)$, i.e. the space $\mathcal{C}^\infty(\Omega)$ topologized through the family of seminorms $p_{K_i,j}(f) = \sup_{x\in K_i, |\alpha|\le j} |\partial^\alpha f(x)|$, where $K_0 \subset K_1 \subset ....K_i \subset ...$ is a countable and exhausting sequence of compact subsets of Ω, provides $\mathcal{G}_E = \mathcal{G}(\Omega)$ ([7, 19]). By Propositions 3.2, 3.4 and Remark 3.5, $\mathcal{G}(\Omega)$ endowed with the sharp topology determined by $\{\mathcal{P}_{K_i,j}\}_{i\in\mathbb{N},j\in\mathbb{N}}$ is a Fréchet $\widetilde{\mathbb{C}}$-module and a topological $\widetilde{\mathbb{C}}$-algebra. Other examples of Fréchet $\widetilde{\mathbb{C}}$-modules which are also topological $\widetilde{\mathbb{C}}$-algebras are given by $\mathcal{G}_E$ when E is $\mathscr{S}(\mathbb{R}^n)$ or $W^{\infty,p}(\mathbb{R}^n)$, $p \in [1, +\infty]$. In this way we construct the algebras $\mathcal{G}_{\mathscr{S}}(\mathbb{R}^n)$ ([15, Definition 2.10]) and $\mathcal{G}_{p,p}(\mathbb{R}^n)$ ([3]) respectively, whose sharp topologies are obtained from $p_k(f) = \sup_{x\in\mathbb{R}^n,|\alpha|\le k}(1 + |x|)^k|\partial^\alpha f(x)|$, $f \in \mathscr{S}(\mathbb{R}^n)$ and $q_k(g) = \max_{|\alpha|\le k} \|\partial^\alpha g\|_p$, $g \in W^{\infty,p}(\mathbb{R}^n)$, with k varying in $\mathbb{N}$. In [14] we prove that a characterization as (3.13) holds for the ideals $\mathcal{N}_{\mathscr{S}(\mathbb{R}^n)} = \mathcal{N}_{\mathscr{S}}(\mathbb{R}^n)$ and $\mathcal{N}_{W^{\infty,p}(\mathbb{R}^n)} = \mathcal{N}_{p,p}(\mathbb{R}^n)$. As a consequence $\mathcal{P}_k$ and $\mathcal{Q}_k$ are ultra-pseudo-norms on $\mathcal{G}_{\mathscr{S}}(\mathbb{R}^n)$ and $\mathcal{G}_{p,p}(\mathbb{R}^n)$ respectively.

We concentrate now on the subalgebra $\mathcal{G}_c(\Omega)$ of generalized functions in $\mathcal{G}(\Omega)$ with compact support. It will turn out that $\mathcal{G}_c(\Omega)$ can be equipped with a strict inductive limit topology, but this procedure requires some preliminary investigations. For technical reason we begin by recalling the basic notions of point value theory in Colombeau algebras [19, 38], which will be used in the sequel.

The set of generalized points $\widetilde{x} \in \widetilde{\Omega}$ is defined as the factor $\Omega_M/\sim$, where $\Omega_M := \{(x_\varepsilon)_\varepsilon \in \Omega^{(0,1]} : \exists N \in \mathbb{N}\ |x_\varepsilon| = O(\varepsilon^{-N})\,\mathrm{as}\,\varepsilon \to 0\}$ and $\sim$ is the equivalence relation given by

$$(x_\varepsilon)_\varepsilon \sim (y_\varepsilon)_\varepsilon \;\Leftrightarrow\; \forall q \in \mathbb{N} \quad |x_\varepsilon - y_\varepsilon| = O(\varepsilon^q).$$

We say that $\widetilde{x} \in \widetilde{\Omega}_c$ if it has a representative $(x_\varepsilon)_\varepsilon$ such that x_ε belongs to a compact set K of Ω for small ε. One can show that the generalized point value of $u \in \mathcal{G}(\Omega)$ at $\widetilde{x} \in \widetilde{\Omega}_c$,

$$u(\widetilde{x}) := [(u_\varepsilon(x_\varepsilon))_\varepsilon]$$

is a well-defined element of $\widetilde{\mathbb{C}}$ and Theorem 1.2.46 in [19] allows the following characterizations of generalized functions in terms of their point values:

$$u = 0 \text{ in } \mathcal{G}(\Omega) \qquad \Leftrightarrow \qquad \forall \widetilde{x} \in \widetilde{\Omega}_c \quad u(\widetilde{x}) = 0 \text{ in } \widetilde{\mathbb{C}}. \tag{3.16}$$

Example 3.7. The Colombeau algebra of compactly supported generalized functions
For $K \Subset \Omega$ we denote by $\mathcal{G}_K(\Omega)$ the space of all generalized functions in $\mathcal{G}(\Omega)$ with support contained in K. Note that $\mathcal{G}_K(\Omega)$ is contained in $\mathcal{G}_{\mathcal{D}_{K'}(\Omega)}$ for all

compact subsets K' of Ω such that $K \subseteq \mathrm{int}(K')$, where $\mathcal{D}_{K'}(\Omega)$ is the space of all smooth functions f with $\mathrm{supp}\, f \subseteq K'$. In fact if $\mathrm{supp}\, u \subseteq K$ we can always find a $\mathcal{D}_{K'}(\Omega)$-moderate representative $(u_\varepsilon)_\varepsilon$ and for all representatives $(u_\varepsilon)_\varepsilon, (u'_\varepsilon)_\varepsilon$ of this type, $(u_\varepsilon - u'_\varepsilon)_\varepsilon \in \mathcal{N}_{\mathcal{D}_{K'}(\Omega)}$. With this choice of $(u_\varepsilon)_\varepsilon$, from $\mathcal{N}_{\mathcal{D}_{K'}(\Omega)} \subseteq \mathcal{N}(\Omega)$ we have that

$$\mathcal{G}_K(\Omega) \to \mathcal{G}_{\mathcal{D}_{K'}(\Omega)} : u \to (u_\varepsilon)_\varepsilon + \mathcal{N}_{\mathcal{D}_{K'}(\Omega)}$$

is a well-defined and injective $\widetilde{\mathbb{C}}$-linear map. Moreover, by $\mathcal{M}_{\mathcal{D}_{K'}(\Omega)} \cap \mathcal{N}(\Omega) \subseteq \mathcal{N}_{\mathcal{D}_{K'}(\Omega)}$, $\mathcal{G}_{\mathcal{D}_{K'}(\Omega)}$ is naturally embedded into $\mathcal{G}(\Omega)$ via

$$\mathcal{G}_{\mathcal{D}_{K'}(\Omega)} \to \mathcal{G}(\Omega) : (u_\varepsilon)_\varepsilon + \mathcal{N}_{\mathcal{D}_{K'}(\Omega)} \to (u_\varepsilon)_\varepsilon + \mathcal{N}(\Omega).$$

In $\mathcal{G}(\Omega)$ the $p_{K,n}$-valuation where $p_{K,n}(f) = \sup_{x\in K, |\alpha|\le n} |\partial^\alpha f(x)|$ is obtained as the valuation of the complex generalized number $\sup_{x\in K,|\alpha|\le n} |\partial^\alpha u(x)| := (\sup_{x\in K,|\alpha|\le n} |\partial^\alpha u_\varepsilon(x)|)_\varepsilon + \mathcal{N}$. Hence for $K, K' \Subset \Omega$, $K \subseteq \mathrm{int}(K')$,

$$\mathrm{v}_{K,n}(u) = \mathrm{v}_{p_{K',n}}(u) \tag{3.17}$$

is a valuation on $\mathcal{G}_K(\Omega)$. More precisely (3.17) does not depend on K' since for any K'_1, K'_2 containing K in their interiors and for any $u \in \mathcal{G}_K(\Omega)$ we have that

$$\mathrm{v}_{p_{K'_1,n}}(u) \ge \inf\left\{\mathrm{v}_{p_{K'_1\setminus \mathrm{int}(K'_1\cap K'_2),n}}(u), \mathrm{v}_{p_{K'_2,n}}(u)\right\} = \mathrm{v}_{p_{K'_2,n}}(u).$$

$\mathcal{G}_K(\Omega)$ with the topology induced by the ultra-pseudo-seminorms $\{\mathcal{P}_{\mathcal{G}_K(\Omega),n}(u) := \mathrm{e}^{-\mathrm{v}_{K,n}(u)}\}_{n\in\mathbb{N}}$ is a locally convex topological $\widetilde{\mathbb{C}}$-module and by construction its topology coincides with the topology induced by any $\mathcal{G}_{\mathcal{D}_{K'}(\Omega)}$ with $K \subseteq \mathrm{int}(K')$. In particular, the $\widetilde{\mathbb{C}}$-module $\mathcal{G}_K(\Omega)$ is separated and by Theorem 1.14 it is metrizable. Finally assume that $u \in \mathcal{G}_{\mathcal{D}_{K'}(\Omega)}$ adheres to $\mathcal{G}_K(\Omega)$. We find a sequence $(u_n)_n \in \mathcal{G}_K(\Omega)$ such that $\mathrm{v}_{p_{K',0}}(u - u_n) \ge n$ for all $n \in \mathbb{N}$. Recall that for all $\widetilde{x} \in \widetilde{V}_{\mathrm{c}}$, where $V = \Omega \setminus K$, the point values $u_n(\widetilde{x})$ are zero in $\widetilde{\mathbb{C}}$ and

$$\mathrm{v}_{\widetilde{\mathbb{C}}}(u(\widetilde{x})) \ge \min\{\mathrm{v}_{p_{K',0}}(u - u_n), \mathrm{v}_{\widetilde{\mathbb{C}}}(u_n(\widetilde{x}))\} = \mathrm{v}_{p_{K',0}}(u - u_n).$$

Consequently $u(\widetilde{x}) = 0$ in $\widetilde{\mathbb{C}}$ and by (3.16) $\mathrm{supp}\, u \subseteq K$. We just proved that $\mathcal{G}_K(\Omega)$ is closed in $\mathcal{G}_{\mathcal{D}_{K'}(\Omega)}$ and since $\mathcal{G}_{\mathcal{D}_{K'}(\Omega)}$ with its sharp topology is complete, Remark 1.30 allows to conclude that $(\mathcal{G}_K(\Omega), \{\mathcal{P}_{\mathcal{G}_K(\Omega),n}\}_{n\in\mathbb{N}})$ *is a Fréchet* $\widetilde{\mathbb{C}}$-*module.*
Note that if $K_1 \subseteq K_2$ then $\mathcal{G}_{K_1}(\Omega) \subseteq \mathcal{G}_{K_2}(\Omega)$ and that $\mathcal{G}_{K_2}(\Omega)$ induces on $\mathcal{G}_{K_1}(\Omega)$ the original topology. By construction $\mathcal{G}_{K_1}(\Omega)$ is closed in $\mathcal{G}_{K_2}(\Omega)$.

Let $(K_n)_{n\in\mathbb{N}}$ be an exhausting sequence of compact subsets of Ω such that $K_n \subseteq K_{n+1}$. Clearly $\mathcal{G}_c(\Omega) = \cup_{n\in\mathbb{N}}\mathcal{G}_{K_n}(\Omega)$. Each $\mathcal{G}_{K_n}(\Omega)$ is a Fréchet $\widetilde{\mathbb{C}}$-module and the assumptions of Definition 1.20, Theorem 1.26 and Theorem 1.32 are satisfied by $\mathcal{G}_n = \mathcal{G}_{K_n}(\Omega)$. Therefore $\mathcal{G}_c(\Omega)$ endowed with the strict inductive limit topology of the sequence $(\mathcal{G}_{K_n}(\Omega))_n$ is a separated and complete locally convex topological $\widetilde{\mathbb{C}}$-module. Obviously this topology is independent of the choice of the covering $(K_n)_n$.
Applying Corollary 1.29 to this context we have that *a sequence* $(u_n)_n$ *of generalized functions with compact support converges to* 0 *in* $\mathcal{G}_c(\Omega)$ *if and only if it is contained in some* $\mathcal{G}_K(\Omega)$ *and convergent to* 0 *there.*

Remark 3.8. Classically [28, Example 7, p.170] the topology on $\mathcal{D}(\Omega)$ is determined by the seminorms

$$p_\theta(f) = \sup_{\alpha\in\mathbb{N}^n} \sup_{x\in\Omega} |\theta_\alpha(x)\partial^\alpha f(x)|,$$

where θ runs through all possible families $\theta = (\theta_\alpha)_{\alpha\in\mathbb{N}^n}$ of continuous functions on Ω with $(\operatorname{supp}\theta_\alpha)_{\alpha\in\mathbb{N}^n}$ locally finite. Easy computations show that

$$\iota_{\mathcal{D}} : \mathcal{G}_{\mathrm{c}}(\Omega) \to \mathcal{G}_{\mathcal{D}(\Omega)} : u \to (u_\varepsilon)_\varepsilon + \mathcal{N}_{\mathcal{D}(\Omega)}, \tag{3.18}$$

where $(u_\varepsilon)_\varepsilon$ is any representative of u with $\mathrm{supp}(u_\varepsilon)$ contained in the same compact set of Ω for all $\varepsilon \in (0,1]$, is well-defined and injective. One may think of endowing $\mathcal{G}_c(\Omega)$ with the locally convex $\widetilde{\mathbb{C}}$-linear topology induced by the sharp topology on $\mathcal{G}_{\mathcal{D}(\Omega)}$ via $\iota_{\mathcal{D}}$. Denoting this topology by $\tau_{\mathcal{D}}$ and the strict inductive limit topology of Example 3.7 by τ, we have that $\tau_{\mathcal{D}}$ is coarser than τ since every embedding $\mathcal{G}_{K_n}(\Omega) \to \mathcal{G}_c(\Omega)$ is continuous for $\tau_{\mathcal{D}}$ on $\mathcal{G}_c(\Omega)$. In detail, taking $K'_n \Subset \Omega$ with $K_n \subseteq \mathrm{int}(K'_n)$ for all $(\theta_\alpha)_\alpha$ there exists $N \in \mathbb{N}$ and $C > 0$ such that the estimate

$$\begin{aligned} \sup_{\alpha\in\mathbb{N}^n} \sup_{x\in\Omega} |\theta_\alpha(x)\partial^\alpha u_\varepsilon(x)| &= \sup_{|\alpha|\le N} \sup_{x\in K'_n} |\theta_\alpha(x)\partial^\alpha u_\varepsilon(x)| \\ &\le C \sup_{|\alpha|\le N, x\in K'_n} |\partial^\alpha u_\varepsilon(x)| = C\, p_{K'_n,N}(u_\varepsilon) \end{aligned} \tag{3.19}$$

holds for every representative of $u \in \mathcal{G}_{K_n}(\Omega)$ with $\mathrm{supp}(u_\varepsilon) \subseteq K'_n$ for all $\varepsilon \in (0,1]$. (3.19) implies

$$\mathcal{P}_\theta(u) \le \mathcal{P}_{\mathcal{G}_{K_n}(\Omega),N}(u), \qquad u \in \mathcal{G}_{K_n}(\Omega)$$

and by Corollary 1.17 guarantees the continuity of the embeddings mentioned above.

In general $\tau_{\mathcal{D}}$ does not coincide with τ. This is shown by the fact that there exist sequences in $\mathcal{G}_c(\mathbb{R}^n)$ which converge to 0 with respect to $\tau_{\mathcal{D}}$ but not with respect to τ. Indeed, for every $\psi \in \mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^n)$ the generalized functions $u_n := (\varepsilon^n\psi(\frac{x}{n}))_\varepsilon + \mathcal{N}(\mathbb{R}^n)$ have compact support and at fixed n

$$\sup_{\alpha\in\mathbb{N}^n} \sup_{x\in\mathbb{R}^n} \big| \varepsilon^n n^{-|\alpha|}\theta_\alpha(x)\partial^\alpha\psi(\frac{x}{n})\big| = O(\varepsilon^n), \quad \text{as}\, \varepsilon \to 0$$

Thus $\mathrm{v}_{p_\theta}(u_n) \ge n \to +\infty$. This means that $(u_n)_n$ is $\tau_{\mathcal{D}}$-convergent to 0. Since $\operatorname{supp} u_n = n \operatorname{supp}\psi$, by Example 3.7 $(u_n)_n$ cannot be τ-convergent to 0.

Example 3.9. The algebra of tempered generalized functions $\mathcal{G}_\tau(\mathbb{R}^n)$
The algebra of tempered generalized functions $\mathcal{G}_\tau(\mathbb{R}^n)$ may be introduced referring to the constructions of [7, 19] as the factor space $\mathcal{E}_\tau(\mathbb{R}^n)/\mathcal{N}_\tau(\mathbb{R}^n)$, where $\mathcal{E}_\tau(\mathbb{R}^n)$ is the algebra of all τ-*moderate* nets $(u_\varepsilon)_\varepsilon \in \mathcal{E}_\tau[\mathbb{R}^n] := \mathcal{O}_M(\mathbb{R}^n)^{(0,1]}$ such that

$$\forall\alpha \in \mathbb{N}^n\, \exists N \in \mathbb{N} \qquad \sup_{x\in\mathbb{R}^n} (1+|x|)^{-N}|\partial^\alpha u_\varepsilon(x)| = O(\varepsilon^{-N}) \qquad \text{as } \varepsilon \to 0 \tag{3.20}$$

and $\mathcal{N}_\tau(\mathbb{R}^n)$ is the ideal of all τ-*negligible* nets $(u_\varepsilon)_\varepsilon \in \mathcal{E}_\tau[\mathbb{R}^n]$ such that

$$\forall\alpha \in \mathbb{N}^n\, \exists N \in \mathbb{N}\, \forall q \in \mathbb{N} \qquad \sup_{x\in\mathbb{R}^n} (1+|x|)^{-N}|\partial^\alpha u_\varepsilon(x)| = O(\varepsilon^q) \text{ as } \varepsilon \to 0. \tag{3.21}$$

Theorem 1.2.25 in [19] shows that $\mathcal{N}_\tau(\mathbb{R}^n)$ coincides with the set of all $(u_\varepsilon)_\varepsilon \in \mathcal{E}_\tau(\mathbb{R}^n)$ whose 0-th derivative satisfies (3.21) i.e. $\exists N \in \mathbb{N}\, \forall q \in \mathbb{N}\ \sup_{x\in\mathbb{R}^n}(1+|x|)^{-N}|u_\varepsilon(x)| = O(\varepsilon^q)$. Moreover for each $\widetilde{x} \in \widetilde{\mathbb{R}^n}$, $u(\widetilde{x}) := [(u_\varepsilon(x_\varepsilon))_\varepsilon]$ is a well-defined element of $\widetilde{\mathbb{C}}$ and the point value characterization

$$\text{(3.22)} \qquad u = 0 \text{ in } \mathcal{G}_\tau(\mathbb{R}^n) \qquad \Leftrightarrow \qquad \forall \widetilde{x} \in \widetilde{\mathbb{R}^n} \quad u(\widetilde{x}) = 0 \text{ in } \widetilde{\mathbb{C}}$$

holds. We present a locally convex $\widetilde{\mathbb{C}}$-linear topology on $\mathcal{G}_\tau(\mathbb{R}^n)$ whose construction involves a countable family of different algebras of generalized functions. We denote by $\mathcal{G}_{\tau,\mathscr{S}}(\mathbb{R}^n)$ the factor algebra $\mathcal{E}_\tau(\mathbb{R}^n)/\mathcal{N}_{\mathscr{S}}(\mathbb{R}^n)$ (c.f. Definition 2.8 [13]) where $\mathcal{N}_{\mathscr{S}}(\mathbb{R}^n) = \mathcal{N}_{\mathscr{S}(\mathbb{R}^n)}$. Inspired by the definition of τ-moderate nets we introduce the set

$$\mathcal{E}^m_N(\mathbb{R}^n) := \{(u_\varepsilon)_\varepsilon \in \mathcal{E}_\tau[\mathbb{R}^n] :\ \exists b \in \mathbb{R} \\ \sup_{x\in\mathbb{R}^n, |\alpha|\le m}(1+|x|)^{-N}|\partial^\alpha u_\varepsilon(x)| = O(\varepsilon^b) \text{ as } \varepsilon \to 0\}$$

and the $\widetilde{\mathbb{C}}$-module $\mathcal{G}^m_{N,\mathscr{S}}(\mathbb{R}^n) := \mathcal{E}^m_N(\mathbb{R}^n)/\mathcal{N}_{\mathscr{S}}(\mathbb{R}^n)$. Thus, setting $\mathcal{G}^m_{\tau,\mathscr{S}}(\mathbb{R}^n) := \cup_{N\in\mathbb{N}}\mathcal{G}^m_{N,\mathscr{S}}(\mathbb{R}^n)$ we have that

$$\mathcal{G}_{\tau,\mathscr{S}}(\mathbb{R}^n) = \bigcap_{m\in\mathbb{N}} \mathcal{G}^m_{\tau,\mathscr{S}}(\mathbb{R}^n) = \bigcap_{m\in\mathbb{N}}\bigcup_{N\in\mathbb{N}} \mathcal{G}^m_{N,\mathscr{S}}(\mathbb{R}^n).$$

We begin by endowing $\mathcal{G}_{\tau,\mathscr{S}}(\mathbb{R}^n)$ with a locally convex $\widetilde{\mathbb{C}}$-linear topology considering $\mathcal{G}_\tau(\mathbb{R}^n)$ only in a second step. Every $\mathcal{G}^m_{N,\mathscr{S}}(\mathbb{R}^n)$ is a locally convex topological $\widetilde{\mathbb{C}}$-module for the ultra-pseudo-seminorm $\mathcal{P}^m_N$ determined by the well-defined valuation

$$\mathrm{v}^m_N(u) := \sup\{b \in \mathbb{R} :\ \sup_{x\in\mathbb{R}^n, |\alpha|\le m}(1+|x|)^{-N}|\partial^\alpha u_\varepsilon(x)| = O(\varepsilon^b) \text{ as } \varepsilon \to 0\}.$$

Hence, we equip $\mathcal{G}^m_{\tau,\mathscr{S}}(\mathbb{R}^n)$ with the inductive limit topology of the sequence $(\mathcal{G}^m_{N,\mathscr{S}}(\mathbb{R}^n))_{N\in\mathbb{N}}$ and we take the initial topology on $\mathcal{G}_{\tau,\mathscr{S}}(\mathbb{R}^n) = \cap_{m\in\mathbb{N}}\mathcal{G}^m_{\tau,\mathscr{S}}(\mathbb{R}^n)$. Finally we topologize $\mathcal{G}_\tau(\mathbb{R}^n)$ through the finest locally convex $\widetilde{\mathbb{C}}$-linear topology such that the map

$$\iota_{\tau,\mathscr{S}} : \mathcal{G}_{\tau,\mathscr{S}}(\mathbb{R}^n) \to \mathcal{G}_\tau(\mathbb{R}^n) : (u_\varepsilon)_\varepsilon + \mathcal{N}_{\mathscr{S}}(\mathbb{R}^n) \to (u_\varepsilon)_\varepsilon + \mathcal{N}_\tau(\mathbb{R}^n)$$

is continuous. The fact that this topology is separated follows from the continuous embedding of $\mathcal{G}_\tau(\mathbb{R}^n)$ into the separated locally convex topological $\widetilde{\mathbb{C}}$-module $\mathrm{L}(\mathcal{G}_{\mathscr{S}}(\mathbb{R}^n), \widetilde{\mathbb{C}})$ studied in [14].

Example 3.10. Regular generalized functions based on E

For any locally convex topological vector space $(E, \{p_i\}_{i\in I})$ the set

$$\text{(3.23)} \quad \mathcal{M}^\infty_E := \{(u_\varepsilon)_\varepsilon \in E^{(0,1]} :\ \exists N \in \mathbb{N}\, \forall i \in I \quad p_i(u_\varepsilon) = O(\varepsilon^{-N}) \text{ as } \varepsilon \to 0\}$$

is a subspace of the set $\mathcal{M}_E$ of E-moderate nets. Therefore the corresponding factor space $\mathcal{G}^\infty_E := \mathcal{M}^\infty_E/\mathcal{N}_E$ is a subspace of $\mathcal{G}_E$ whose elements are called *regular generalized functions based on E*. When E is in addition a topological algebra, i.e. estimate (3.15) is satisfied, $\mathcal{G}_E$ and $\mathcal{G}^\infty_E$ are both algebras. We

want to equip $\mathcal{G}_E^\infty$ with a suitable $\widetilde{\mathbb{C}}$-linear topology and discuss some examples concerning Colombeau algebras of regular generalized functions.

First of all since $\mathcal{G}_E^\infty \subseteq \mathcal{G}_E$ the sharp topology induced by $\mathcal{G}_E$ on $\mathcal{G}_E^\infty$ gives the structure of a separated locally convex topological $\widetilde{\mathbb{C}}$-module to $\mathcal{G}_E^\infty$. The ultra-pseudo-seminorms obtained in this way are the original $\mathcal{P}_i$ on $\mathcal{G}_E$ restricted to $\mathcal{G}_E^\infty$.

The moderateness properties of $\mathcal{M}_E^\infty$ allow us to define the valuation $\mathrm{v}_E^\infty : \mathcal{M}_E^\infty \to (-\infty, +\infty]$ as

$$\mathrm{v}_E^\infty((u_\varepsilon)_\varepsilon) = \sup\{b \in \mathbb{R} : \ \forall i \in I \ p_i(u_\varepsilon) = O(\varepsilon^b) \text{ as } \varepsilon \to 0\}$$

which can be obviously extended to $\mathcal{G}_E^\infty$. This yields the existence of the ultra-pseudo-norm

$$\mathcal{P}_E^\infty(u) := \mathrm{e}^{-\mathrm{v}_E^\infty(u)}$$

on $\mathcal{G}_E^\infty$. Since for all $i \in I$ and $u \in \mathcal{G}_E^\infty$, $\mathrm{v}_{p_i}(u) \geq \mathrm{v}_E^\infty(u)$, the topology $\tau_\sharp^\infty$ determined by $\mathcal{P}_E^\infty$ on $\mathcal{G}_E^\infty$ is finer than the topology induced by $\mathcal{G}_E$.

Adapting Proposition 3.4 to this situation one easily shows that when E is a locally convex topological vector space with a countable base of neighborhoods of the origin, $\mathcal{G}_E^\infty$ with the topology $\tau_\sharp^\infty$ is complete. In fact assuming that E is topologized through an increasing sequence of seminorms $\{p_k\}_{k\in\mathbb{N}}$, if $(u_n)_n$ is a Cauchy sequence in $\mathcal{G}_E^\infty$ then we can extract a subsequence $(u_{n_k})_k$ such that $\mathrm{v}_E^\infty(u_{n_{k+1}} - u_{n_k}) > k$. This means that $\mathrm{v}_{p_i}(u_{n_{k+1}} - u_{n_k}) > k$ for all $i \in \mathbb{N}$ and that there exists a decreasing sequence $\varepsilon_k \searrow 0$, $\varepsilon_k \leq 2^{-k}$ such that $p_k(u_{n_{k+1},\varepsilon} - u_{n_k,\varepsilon}) \leq \varepsilon^k$ for all $\varepsilon \in (0, \varepsilon_k)$. Defining $(h_{k,\varepsilon})_\varepsilon$ as in the proof of Proposition 3.4, $(h_{k,\varepsilon})_\varepsilon \in \mathcal{M}_E^\infty$ and for all $k' \leq k$ we have that $p_{k'}(h_{k,\varepsilon}) \leq \varepsilon^k$ on the interval $(0, 1]$. As a consequence $u_\varepsilon = u_{n_0,\varepsilon} + \sum_{k=0}^\infty h_{k,\varepsilon}$ is an element of $\mathcal{M}_E^\infty$ and for all $\overline{k} \geq 1$, $k \leq \overline{k}$ and $\varepsilon \in (0, \varepsilon_{\overline{k}-1})$

$$p_k(u_{n_{\overline{k}},\varepsilon} - u_\varepsilon) \leq p_{\overline{k}}(u_{n_{\overline{k}},\varepsilon} - u_\varepsilon) \leq c\varepsilon^{\overline{k}-1}.$$

When $k > \overline{k}$ we may write

$$p_k(u_{n_{\overline{k}},\varepsilon} - u_\varepsilon) \leq p_k(u_{n_{\overline{k}},\varepsilon} - u_{n_k,\varepsilon}) + p_k(u_{n_{k,\varepsilon}} - u_\varepsilon)$$

where as before $p_k(u_{n_{k,\varepsilon}} - u_\varepsilon) = O(\varepsilon^{k-1}) = O(\varepsilon^{\overline{k}-1})$ and $p_k(u_{n_{\overline{k}},\varepsilon} - u_{n_k,\varepsilon}) = O(\varepsilon^{\overline{k}}) = O(\varepsilon^{\overline{k}-1})$ using the assumption $\mathrm{v}_E^\infty(u_{n_{k+1}} - u_{n_k}) > k$ and a telescope sum argument. In this way we obtain that $\mathrm{v}_E^\infty(u_{n_{\overline{k}},\varepsilon} - u_\varepsilon) \geq \overline{k} - 1$ and therefore $u_{n_k} \to u$ in $(\mathcal{G}_E^\infty, \tau_\sharp^\infty)$.
In conclusion we can say that if E has a countable base of neighborhoods of the origin then the associated space $\mathcal{G}_E^\infty$ of regular generalized functions is a complete and ultra-pseudo-normed $\widetilde{\mathbb{C}}$-module.

Example 3.11. The Colombeau algebra of $\mathscr{S}$-regular generalized functions

A concrete example of $\mathcal{G}_E^\infty$ is given by the Colombeau algebra of $\mathscr{S}$-regular generalized functions $\mathcal{G}_{\mathscr{S}}^\infty(\mathbb{R}^n)$ introduced in [13, 15], whose definition is precisely $\mathcal{G}_E^\infty$ with $E = \mathscr{S}(\mathbb{R}^n)$. In this case we have that $\mathrm{v}_{\mathscr{S}(\mathbb{R}^n)}^\infty(u) := \sup\{b \in \mathbb{R} :$

$\forall k \in \mathbb{N} \quad \sup_{x\in\mathbb{R}^n,|\alpha|\le k}(1+|x|)^k|\partial^\alpha u_\varepsilon(x)| = O(\varepsilon^b)\}$ and since $\mathcal{P}^\infty_{\mathscr{S}(\mathbb{R}^n)}(uv) \le \mathcal{P}^\infty_{\mathscr{S}(\mathbb{R}^n)}(u)\mathcal{P}^\infty_{\mathscr{S}(\mathbb{R}^n)}(v)$ it turns out that $\mathcal{G}^\infty_{\mathscr{S}}(\mathbb{R}^n)$ is a topological $\widetilde{\mathbb{C}}$-algebra and a complete ultra-pseudo-normed $\widetilde{\mathbb{C}}$-module.

Example 3.12. The Colombeau algebra $\mathcal{G}^\infty(\Omega)$

We recall that the *Colombeau algebra of regular generalized functions* is the set $\mathcal{G}^\infty(\Omega)$ of all $u \in \mathcal{G}(\Omega)$ having a representative $(u_\varepsilon)_\varepsilon$ satisfying the following condition

$$\forall K \Subset \Omega\ \exists N \in \mathbb{N}\ \forall \alpha \in \mathbb{N}^n \qquad \sup_{x\in K}|\partial^\alpha u_\varepsilon(x)| = O(\varepsilon^{-N}) \text{ as } \varepsilon \to 0. \tag{3.24}$$

(3.24) defines the subset $\mathcal{E}^\infty_M(\Omega)$ of the set of moderate nets $\mathcal{E}_M(\Omega)$ and determines $\mathcal{G}^\infty(\Omega)$ as the factor $\mathcal{E}^\infty_M(\Omega)/\mathcal{N}(\Omega)$. $\mathcal{G}^\infty(\Omega)$ can be seen as the intersection $\cap_{K\Subset\Omega}\mathcal{G}^\infty(K)$ where $\mathcal{G}^\infty(K)$ is the space of all $u \in \mathcal{G}(\Omega)$ such that there exists a representative $(u_\varepsilon)_\varepsilon$ satisfying the condition

$$\exists N \in \mathbb{N}\ \forall \alpha \in \mathbb{N}^n \qquad \sup_{x\in K}|\partial^\alpha u_\varepsilon(x)| = O(\varepsilon^{-N}) \quad \text{as } \varepsilon \to 0. \tag{3.25}$$

Let us choose an exhausting sequence $K = K_0 \subset K_1 \subset K_2 ...$ of compact subsets of Ω. We equip $\mathcal{G}^\infty(K)$ with the locally convex $\widetilde{\mathbb{C}}$-linear topology determined by the usual ultra-pseudo-seminorms $\mathcal{P}_i(u) = \mathrm{e}^{-\mathrm{v}_{p_i}(u)}$ on $\mathcal{G}(\Omega)$ where $p_i(u_\varepsilon) = \sup_{x\in K_i,|\alpha|\le i}|\partial^\alpha u_\varepsilon(x)|$, and the $\mathcal{G}^\infty(K)$-ultra-pseudo-seminorm $\mathcal{P}_{\mathcal{G}^\infty(K)}$ defined via the valuation

$$\mathrm{v}_{\mathcal{G}^\infty(K)}(u) = \sup\{b \in \mathbb{R} :\ \forall \alpha \in \mathbb{N}^n \qquad \sup_{x\in K}|\partial^\alpha u_\varepsilon(x)| = O(\varepsilon^b)\}. \tag{3.26}$$

It is clear that with this topology $\mathcal{G}^\infty(K)$ is separated and metrizable.

We want to prove that $\mathcal{G}^\infty(K)$ is complete. As in the proof of Proposition 3.4 and the reasoning concerning $\mathcal{G}^\infty_E$ in Example 3.10, if $(u_n)_n$ is a Cauchy sequence in $\mathcal{G}^\infty(K)$ we can extract a subsequence $(u_{n_j})_j$ and a sequence $\varepsilon_j \searrow 0$, $\varepsilon_j \le 2^{-j}$ such that for all $j \in \mathbb{N}$, $\mathrm{v}_{\mathcal{G}^\infty(K)}(u_{n_{j+1},\varepsilon} - u_{n_j,\varepsilon}) > j$ and $p_j(u_{n_{j+1},\varepsilon} - u_{n_j,\varepsilon}) \le \varepsilon^j$ on $(0,\varepsilon_j)$. Define the net $(h_{j,\varepsilon})_\varepsilon$ as $u_{n_{j+1},\varepsilon} - u_{n_j,\varepsilon}$ on the interval $(0,\varepsilon_j)$ and 0 outside. By construction $(h_{j,\varepsilon})_\varepsilon$ satisfies (3.25) and by Proposition 3.4 $u_\varepsilon := u_{n_0,\varepsilon} + \sum_{j=0}^\infty h_{j,\varepsilon}$ belongs to $\mathcal{E}_M(\Omega)$. More precisely for all $\alpha \in \mathbb{N}^n$

$$\begin{aligned}\sup_{x\in K}|\partial^\alpha u_\varepsilon(x)| &\le \sup_{x\in K}|\partial^\alpha u_{n_0,\varepsilon}(x)| + \sum_{j=0}^{|\alpha|}\sup_{x\in K,|\beta|\le|\alpha|}|\partial^\beta h_{j,\varepsilon}(x)| + \sum_{j=|\alpha|+1}^\infty p_j(h_{j,\varepsilon})\\ &\le \sup_{x\in K}|\partial^\alpha u_{n_0,\varepsilon}(x)| + \sum_{j=0}^{|\alpha|}\sup_{x\in K,|\beta|\le|\alpha|}|\partial^\beta h_{j,\varepsilon}(x)| + \sum_{j=|\alpha|+1}^\infty \frac{1}{2^j},\end{aligned}$$

where $\sup_{x\in K,|\beta|\le|\alpha|}|\partial^\beta h_{j,\varepsilon}(x)| = O(1)$ for all j and α. It follows that $(u_\varepsilon)_\varepsilon + \mathcal{N}(\Omega) \in \mathcal{G}^\infty(K)$ and adapting the estimates in the proof of Proposition 3.4 to our situation we obtain that for all $\overline{\jmath} \ge 1$

$$p_{\overline{\jmath}}(u_{n_{\overline{\jmath}},\varepsilon} - u_\varepsilon) = O(\varepsilon^{\overline{\jmath}-1}) \quad \text{as } \varepsilon \to 0. \tag{3.27}$$

As a consequence $\sup_{x\in K,|\alpha|\le\overline{\jmath}}|\partial^\alpha(u_{n_{\overline{\jmath}},\varepsilon}-u_\varepsilon)(x)| = O(\varepsilon^{\overline{\jmath}-1})$. If $j\ge\overline{\jmath}$ the assumption $\mathrm{v}_{\mathcal{G}^\infty(K)}(u_{n_{j+1},\varepsilon}-u_{n_j,\varepsilon})>j$ leads to

(3.28)
$$\begin{aligned}
&\sup_{x\in K,|\alpha|\le j}|\partial^\alpha(u_{n_{\overline{\jmath}},\varepsilon}-u_\varepsilon)(x)|\\
&\qquad\le \sup_{x\in K,|\alpha|\le j}|\partial^\alpha(u_{n_{\overline{\jmath}},\varepsilon}-u_{n_j,\varepsilon})(x)| + \sup_{x\in K_j,|\alpha|\le j}|\partial^\alpha(u_{n_j,\varepsilon}-u_\varepsilon)(x)|\\
&\qquad= O(\varepsilon^{\overline{\jmath}})+O(\varepsilon^{j-1}) = O(\varepsilon^{\overline{\jmath}-1}).
\end{aligned}$$

(3.28) shows that $\mathrm{v}_{\mathcal{G}^\infty(K)}(u_{n_{\overline{\jmath}},\varepsilon}-u_\varepsilon)\ge\overline{\jmath}-1$ and combined with (3.27) yields that u_{n_j} is convergent to u in $\mathcal{G}^\infty(K)$. In this way $\mathcal{G}^\infty(K)$ is a Fréchet $\widetilde{\mathbb{C}}$-module.

By definition $\mathcal{G}^\infty(K')$ is a $\widetilde{\mathbb{C}}$-submodule of $\mathcal{G}^\infty(K)$ when $K\subseteq K'$ and noting that $\mathcal{P}_{\mathcal{G}^\infty(K)}(u)\le\mathcal{P}_{\mathcal{G}^\infty(K')}(u)$ for all $u\in\mathcal{G}^\infty(K')$, the topology on $\mathcal{G}^\infty(K')$ is finer than the topology induced by $\mathcal{G}^\infty(K)$ on $\mathcal{G}^\infty(K')$. We are in the situation of Proposition 1.34. Hence $\mathcal{G}^\infty(\Omega)$ equipped with the initial topology for the injections $\mathcal{G}^\infty(\Omega)\to\mathcal{G}^\infty(K)$ is a complete locally convex topological $\widetilde{\mathbb{C}}$-module. More precisely since for every $\mathcal{P}_i$ as above the estimate $\mathcal{P}_i(u)\le\mathcal{P}_{\mathcal{G}^\infty(K_i)}(u)$ holds on $\mathcal{G}^\infty(\Omega)$, the initial topology on $\mathcal{G}^\infty(\Omega)$ is determined by a countable family of ultra-pseudo-seminorms and then $\mathcal{G}^\infty(\Omega)$ itself is a Fréchet $\widetilde{\mathbb{C}}$-module. This topology on $\mathcal{G}^\infty(\Omega)$ is finer than the sharp topology induced by $\mathcal{G}(\Omega)$ on $\mathcal{G}^\infty(\Omega)$ and makes the multiplication of generalized functions in $\mathcal{G}^\infty(\Omega)$ continuous.

Note that choosing $E=\mathcal{E}(\Omega)$ in Example 3.10 we can construct the algebra $\mathcal{G}^\infty_{\mathcal{E}(\Omega)}$. Obviously $\mathcal{G}^\infty_{\mathcal{E}(\Omega)}\subseteq\mathcal{G}^\infty(\Omega)$ but they do not coincide since the estimates which concern the representatives in $\mathcal{G}^\infty_{\mathcal{E}(\Omega)}$ require the same power of ε for all derivatives and all compact sets K. Finally $\mathcal{P}_{\mathcal{G}^\infty(K)}(u)\le\mathcal{P}^\infty_{\mathcal{E}(\Omega)}(u)$ for all $K\Subset\Omega$ and $u\in\mathcal{G}^\infty_{\mathcal{E}(\Omega)}$.

Example 3.13. The Colombeau algebra $\mathcal{G}^\infty_{\mathrm{c}}(\Omega)$

$\mathcal{G}^\infty_{\mathrm{c}}(\Omega)$ denotes the algebra of generalized functions in $\mathcal{G}^\infty(\Omega)$ which have compact support. We want to endow this space with a $\widetilde{\mathbb{C}}$-linear topology. By the previous considerations each $\mathcal{G}^\infty_{\mathcal{D}_K(\Omega)}$, $K\Subset\Omega$, is a complete ultra-pseudo-normed $\widetilde{\mathbb{C}}$-module with valuation

$$\mathrm{v}^\infty_{\mathcal{D}_K(\Omega)}(u)=\sup\{b\in\mathbb{R}:\ \forall\alpha\in\mathbb{N}^n\ \sup_{x\in K}|\partial^\alpha u_\varepsilon(x)|=O(\varepsilon^b)\}.$$

Note that $\mathrm{v}_{\mathcal{G}^\infty(K)}$ defined in (3.26) and $\mathrm{v}^\infty_{\mathcal{D}_K(\Omega)}$ coincide on $\mathcal{G}^\infty_{\mathcal{D}_K(\Omega)}$. Repeating the reasoning of Example 3.7, $\mathcal{G}^\infty_K(\Omega)$, the space of all generalized functions in $\mathcal{G}^\infty(\Omega)$ with support contained in K, is contained in $\mathcal{G}^\infty_{\mathcal{D}_{K'}(\Omega)}$, for every compact K' containing K in its interior. This inclusion is given by

$$\mathcal{G}^\infty_K(\Omega)\to\mathcal{G}^\infty_{\mathcal{D}_{K'}(\Omega)}: u\to(u_\varepsilon)_\varepsilon+\mathcal{N}_{\mathcal{D}_{K'}(\Omega)},$$

where we choose a representative $(u_\varepsilon)_\varepsilon$ of u in $\mathcal{E}^\infty_M(\Omega)\cap\mathcal{M}_{\mathcal{D}_{K'}(\Omega)}$. It is clear that $\mathcal{G}^\infty_{\mathcal{D}_{K'}(\Omega)}$ is contained in $\mathcal{G}^\infty(\Omega)$. Since for every $K'_1,K'_2\Subset\Omega$ with $K\subseteq\mathrm{int}(K'_1)\cap\mathrm{int}(K'_2)$ and $u\in\mathcal{G}^\infty_K(\Omega)$ we have that $\mathrm{v}^\infty_{\mathcal{D}_{K'_2}(\Omega)}(u)=\mathrm{v}^\infty_{\mathcal{D}_{K'_1}(\Omega)}(u)$, we may define

the valuation v_K^∞ on $\mathcal{G}_K^\infty(\Omega)$ as $\mathrm{v}_K^\infty(u) = \mathrm{v}^\infty_{\mathcal{D}_{K'}(\Omega)}(u)$ where $K \subseteq \mathrm{int}(K') \Subset \Omega$. In this way we can equip $\mathcal{G}_K^\infty(\Omega)$ with the $\widetilde{\mathbb{C}}$-linear topology determined by the ultra-pseudo-norm $\mathcal{P}_{\mathcal{G}_K^\infty(\Omega)}(u) = \mathrm{e}^{-\mathrm{v}_K^\infty(u)}$. Note that this topology is finer than the one induced by $\mathcal{G}_K(\Omega)$ on $\mathcal{G}_K^\infty(\Omega)$.

Since the topology considered on $\mathcal{G}^\infty_{\mathcal{D}_{K'}(\Omega)}$ is finer than the topology induced by $\mathcal{G}_{\mathcal{D}_{K'}}(\Omega)$ on $\mathcal{G}^\infty_{\mathcal{D}_{K'}(\Omega)}$ and $\mathcal{G}_K(\Omega)$ is a closed subset of $\mathcal{G}_{\mathcal{D}_{K'}(\Omega)}$ we have that $\mathcal{G}_K^\infty(\Omega)$ is closed in $\mathcal{G}^\infty_{\mathcal{D}_{K'}(\Omega)}$. Hence $\mathcal{G}_K^\infty(\Omega)$ is complete for the topology defined by $\mathcal{P}_{\mathcal{G}_K^\infty(\Omega)}$. In analogy with the non-regular context examined before, $\mathcal{G}^\infty_{K_2}(\Omega)$ induces on $\mathcal{G}^\infty_{K_1}(\Omega)$ the original topology if $K_1 \subseteq K_2$ and $\mathcal{G}^\infty_{K_1}(\Omega)$ is closed in $\mathcal{G}^\infty_{K_2}(\Omega)$.

At this point given an exhausting sequence $K_0 \subset K_1 \subset K_2 \ldots$. of compact sets, the strict inductive limit procedure equips $\mathcal{G}_\mathrm{c}^\infty(\Omega) = \cup_{n\in\mathbb{N}}\mathcal{G}^\infty_{K_n}(\Omega)$ with a complete and separated locally convex $\widetilde{\mathbb{C}}$-linear topology. Denoting the topologies on $\mathcal{G}_{K_n}(\Omega)$ and $\mathcal{G}^\infty_{K_n}(\Omega)$ by τ_n and τ_n^∞ respectively and the inductive limit topologies on $\mathcal{G}_c(\Omega)$ and $\mathcal{G}_\mathrm{c}^\infty(\Omega)$ by τ and τ^∞ respectively, we obtain that τ^∞ is the finest locally convex $\widetilde{\mathbb{C}}$-linear topology such that the embeddings $(\mathcal{G}^\infty_{K_n}(\Omega), \tau_n^\infty) \to (\mathcal{G}_\mathrm{c}^\infty(\Omega), \tau^\infty)$ are continuous. Moreover since the embedding maps $(\mathcal{G}^\infty_{K_n}(\Omega), \tau_n^\infty) \to (\mathcal{G}_{K_n}(\Omega), \tau_n) \to (\mathcal{G}_\mathrm{c}(\Omega), \tau)$ are continuous, the topology τ^∞ on $\mathcal{G}_\mathrm{c}^\infty(\Omega)$ is finer than the topology induced by $\mathcal{G}_\mathrm{c}(\Omega)$.

3.2 Continuity of a $\widetilde{\mathbb{C}}$-linear map $T : \mathcal{G}_E \to \mathcal{G}$

We already argued on the continuity of a $\widetilde{\mathbb{C}}$-linear map between locally convex topological $\widetilde{\mathbb{C}}$-modules in Subsection 1.2, Theorem 1.16 and Corollary 1.17. Here we focus our attention on $\widetilde{\mathbb{C}}$-linear maps where at least the domain is of a space of generalized functions $\mathcal{G}_E$ over E. In particular, we investigate the relationships between $\widetilde{\mathbb{C}}$-linearity and continuity with respect to the sharp topology by means of some examples. Before proceeding we recall that given locally convex topological vector spaces $(E, \{p_i\}_{i\in I})$ and $(F, \{q_j\}_{j\in J})$, by (1.9) a $\widetilde{\mathbb{C}}$-linear map $T : \mathcal{G}_E \to \mathcal{G}_F$ is continuous for the corresponding sharp topologies ("is sharp continuous" for short) if and only if for all $j \in J$

$$\mathcal{Q}_j(Tu) \le C \max_{i\in I_0} \mathcal{P}_i(u)$$

for some finite subset I_0 of I, or in terms of valuations $\mathrm{v}_{q_j}(Tu) \ge -\log C + \min_{i\in I_0} \mathrm{v}_{p_i}(u)$. In the following remark we discuss some examples of $\widetilde{\mathbb{C}}$-linear maps which are continuous.

Remark 3.14.

(i) When E is a normed space with $\dim E = n < \infty$, every $\widetilde{\mathbb{C}}$-linear map T from $\mathcal{G}_E$ into a locally convex topological $\widetilde{\mathbb{C}}$-module $\mathcal{G}$ is continuous.

(ii) We say that a $\widetilde{\mathbb{C}}$-linear map $T : \mathcal{G}_E \to \mathcal{G}_F$, where E and F are locally convex topological vector spaces, has a representative $t : E \to F$ if t maps moderate nets into moderate nets and negligible nets into negligible nets, i.e. $(u_\varepsilon)_\varepsilon \in \mathcal{M}_E$

implies $(tu_\varepsilon)_\varepsilon \in \mathcal{M}_F$ and $(u_\varepsilon)_\varepsilon \in \mathcal{N}_E$ implies $(tu_\varepsilon)_\varepsilon \in \mathcal{N}_F$, and $Tu = [(tu_\varepsilon)_\varepsilon]$ for all $u \in \mathcal{G}_E$.

Any linear and continuous map $t : E \to F$ defines the $\widetilde{\mathbb{C}}$-linear and sharp continuous map $T : \mathcal{G}_E \to \mathcal{G}_F : u \to [(tu_\varepsilon)_\varepsilon]$. In fact, since t is continuous T is well-defined and sharp continuous and finally the $\mathbb{C}$-linearity of t yields the $\widetilde{\mathbb{C}}$-linearity of T.

(iii) The sharp continuity of a $\widetilde{\mathbb{C}}$-linear map T with representative $t : E \to \mathbb{C}$ does not guarantee the continuity of the representative. Let $(E, \{p_i\}_{i\in I})$ be a non-bornological locally convex topological vector space and $t : E \to \mathbb{C}$ a linear bounded map which is not continuous. To provide an example of such a map T we consider the space of regular generalized functions based on E and we slightly modify the corresponding definition by requiring uniform estimates in the interval $(0,1]$. In other words we introduce $\underline{\mathcal{G}}^\infty_E \subseteq \mathcal{G}^\infty_E$ by factorizing $\{(u_\varepsilon)_\varepsilon \in E^{(0,1]} :\ \exists N \in \mathbb{N}\, \forall i \in I\ \sup_{\varepsilon\in(0,1]} \varepsilon^N p_i(u_\varepsilon) < \infty\}$ with respect to $\{(u_\varepsilon)_\varepsilon \in E^{(0,1]} :\ \forall i \in I\, \forall q \in \mathbb{N}\ \sup_{\varepsilon\in(0,1]} \varepsilon^{-q} p_i(u_\varepsilon) < \infty\}$. If $(u_\varepsilon)_\varepsilon$ is a representative of $u \in \underline{\mathcal{G}}^\infty_E$ then for some $N \in \mathbb{N}$ the set $\{\varepsilon^N u_\varepsilon,\ \varepsilon \in (0,1]\}$ is bounded in E and from the boundedness of t we have that $|t(\varepsilon^N u_\varepsilon)| = \varepsilon^N |t(u_\varepsilon)| \le c$. This means that $T : \underline{\mathcal{G}}^\infty_E \to \widetilde{\mathbb{C}} : u \to [(tu_\varepsilon)_\varepsilon]$ is well-defined and sharp continuous since $\mathrm{v}_{\widetilde{\mathbb{C}}}(Tu) \ge \mathrm{v}^\infty_E(u)$.

Take now a pairing of vector spaces (E, F, b) and endow E with the weak topology $\sigma(E,F)$. It is clear that for each $y \in F$ the map $E \to \mathbb{C} : x \to \mathsf{b}(x,y)$ is continuous and from (ii) in Remark 3.14 $\mathsf{b}(\cdot, y) : \mathcal{G}_E \to \widetilde{\mathbb{C}} : u \to \mathsf{b}(u,y) := [(\mathsf{b}(u_\varepsilon, y))_\varepsilon]$ is sharp continuous.

Proposition 3.15. *Let (E, F, b) be a pairing and let E be equipped with the weak topology $\sigma(E,F)$. If $T : \mathcal{G}_E \to \widetilde{\mathbb{C}}$ is a sharp continuous $\widetilde{\mathbb{C}}$-linear map with a representative $t : E \to \mathbb{C}$ then there exists $y \in F$ such that $Tu = \mathsf{b}(u,y)$ for all $u \in \mathcal{G}_E$.*

The proof of Proposition 3.15 needs a preparatory lemma.

Lemma 3.16. *Under the assumptions of Proposition 3.15 on E and F, if $T : \mathcal{G}_E \to \widetilde{\mathbb{C}}$ is a $\widetilde{\mathbb{C}}$-linear and sharp continuous map then there exists $\{y_i\}_{i=1}^N \subseteq F$ such that*

$$\bigcap_{i=1}^N \ker \mathsf{b}(\cdot, y_i) \subseteq \ker T. \tag{3.29}$$

Proof. Since T is continuous at the origin, there exists $\{y_i\}_{i=1}^N \subseteq F$ and $\eta > 0$ such that $\max_{i=1}^N |\mathsf{b}(u, y_i)|_{\mathrm{e}} \le \eta$ implies $|Tu|_{\mathrm{e}} \le 1$. Now if $u \in \bigcap_{i=1}^N \ker \mathsf{b}(\cdot, y_i)$ then $|\mathsf{b}(u, y_i)|_{\mathrm{e}} = 0$ for all $i = 1, ..., N$ and the same holds for $w = [(\varepsilon^{-a})_\varepsilon]u$ where a is an arbitrary real number. Thus $|Tw|_{\mathrm{e}} = \mathrm{e}^a |Tu|_{\mathrm{e}} \le 1$ which implies $|Tu|_{\mathrm{e}} \le \mathrm{e}^{-a}$ for all a and therefore $u \in \ker T$. $\square$

Proof of Proposition 3.15. By Lemma 3.16 we know that there exists a finite number of $y_1, y_2, ..., y_N$ in F such that (3.29) holds. Let L be the $\widetilde{\mathbb{C}}$-linear map

from $\mathcal{G}_E$ into $\widetilde{\mathbb{C}}^N$ given by $Lu = (\mathsf{b}(u, y_1), \mathsf{b}(u, y_2), ..., \mathsf{b}(u, y_N))$. The inclusion (3.29) allows us to define $S : L(\mathcal{G}_E) \to \widetilde{\mathbb{C}} : (\mathsf{b}(u, y_1), \mathsf{b}(u, y_2), ..., \mathsf{b}(u, y_N)) \to Tu$. Consider now the subset $V := \{(\mathsf{b}(u, y_1), \mathsf{b}(u, y_2), ..., \mathsf{b}(u, y_N)),\ u \in E\}$ of $\mathbb{C}^N$. By (3.29) the map $s : V \to \mathbb{C} : (\mathsf{b}(u, y_1), \mathsf{b}(u, y_2), ..., \mathsf{b}(u, y_N)) \to tu$ is a well-defined representative of S and it can be obviously extended to a linear map $s' : \mathbb{C}^N \to \mathbb{C}$. This means that if $(e_1, e_2, ..., e_N)$ is the canonical basis of $\mathbb{C}^N$ and $s'(e_i) = \lambda_i$, $i = 1, ..., N$ then we have, for all $u \in \mathcal{G}_E$, that

$$\begin{aligned}
Tu &= S(\mathsf{b}(u, y_1), \mathsf{b}(u, y_2), ..., \mathsf{b}(u, y_N)) = [(s(\mathsf{b}(u_\varepsilon, y_1), \mathsf{b}(u_\varepsilon, y_2), ...\mathsf{b}(u_\varepsilon, y_N)))_\varepsilon] \\
&= \left[\left(\sum_{i=1}^N \lambda_i \mathsf{b}(u_\varepsilon, y_i)\right)_\varepsilon\right] = \left[\left(\mathsf{b}(u_\varepsilon, \sum_{i=1}^N \lambda_i y_i)\right)_\varepsilon\right] = \mathsf{b}\Big(u, \sum_{i=1}^N \lambda_i y_i\Big),
\end{aligned}$$

where $\sum_{i=1}^N \lambda_i y_i \in F$. $\square$

In conclusion Proposition 3.15 combined with (ii) in Remark 3.14 and the classical results on pairings and continuity leads to the following statement: under the assumptions of Proposition 3.15, $T : \mathcal{G}_E \to \widetilde{\mathbb{C}}$ is sharp continuous if and only if there exists $y \in F$ such that $T = \mathsf{b}(\cdot, y)$ if and only if the representative $t : E \to \mathbb{C}$ is continuous.

3.3 The topological dual $Ł(\mathcal{G}_E, \widetilde{\mathbb{C}})$ when E is a normed space

The space of generalized functions $\mathcal{G}_E$ has a simple and interesting topological structure when E is a normed space and we consider the ultra-pseudo-norm $\|u\|_{\mathcal{G}_E} := \mathrm{e}^{-\mathrm{v}_{\|\cdot\|_E}(u)}$. In Section 2 we equipped $Ł(\mathcal{G}_E, \widetilde{\mathbb{C}})$ with three topologies: the weak topology $\sigma(Ł(\mathcal{G}_E, \widetilde{\mathbb{C}}), \mathcal{G}_E)$, the strong topology $\beta(Ł(\mathcal{G}_E, \widetilde{\mathbb{C}}), \mathcal{G}_E)$ and the polar topology $\beta_b(Ł(\mathcal{G}_E, \widetilde{\mathbb{C}}), \mathcal{G}_E)$. The ultra-pseudo-norm introduced on $\mathcal{G}_E$ defines, as in the classical theory of normed spaces, a corresponding ultra-pseudo-norm on $Ł(\mathcal{G}_E, \widetilde{\mathbb{C}})$ adding another $\widetilde{\mathbb{C}}$-linear topology to the list above. For the sake of generality we begin to discuss this topic in the context of an ultra-pseudo-normed $\widetilde{\mathbb{C}}$-module $\mathcal{G}$.

Proposition 3.17. *Let $\mathcal{G}$ be a topological $\widetilde{\mathbb{C}}$-module with topology determined by an ultra-pseudo-norm $\mathcal{P}$. The map $\mathcal{P}_{Ł(\mathcal{G}, \widetilde{\mathbb{C}})}$ defined on $Ł(\mathcal{G}, \widetilde{\mathbb{C}})$ by*

$$\mathcal{P}_{Ł(\mathcal{G}, \widetilde{\mathbb{C}})}(T) = \inf\{C > 0 : \forall u \in \mathcal{G} \quad |Tu|_\mathrm{e} \le C\,\mathcal{P}(u)\ \} \tag{3.30}$$

is an ultra-pseudo-norm on $Ł(\mathcal{G}, \widetilde{\mathbb{C}})$ and it coincides with $\sup\limits_{\mathcal{P}(u)=1} |Tu|_\mathrm{e}$.

Proof. Since it is immediate, we do not prove that (3.30) has the properties which characterize an ultra-pseudo-norm. We note that when $u_0 \neq 0$ in $\mathcal{G}$ the element $[(\varepsilon^{\log \mathcal{P}(u_0)})_\varepsilon]u_0$ belongs to the set of $U := \{u \in \mathcal{G} :\ \mathcal{P}(u) = 1\}$. Hence

$$\begin{aligned}
\sup_{\mathcal{P}(u)=1} |Tu|_\mathrm{e} &= \inf\{C > 0 :\ \forall u \in U \quad |Tu|_\mathrm{e} \le C\} \\
&= \inf\{C > 0 :\ \forall u \in \mathcal{G} \quad |Tu|_\mathrm{e} \le C\mathcal{P}(u)\}.
\end{aligned}$$

$\square$

Remark 3.18. Denoting the topology on $\mathrm{L}(\mathcal{G},\widetilde{\mathbb{C}})$ obtained via $\mathcal{P}_{\mathrm{L}(\mathcal{G},\widetilde{\mathbb{C}})}$ by τ we can write the chain of relationships

$$\sigma(\mathrm{L}(\mathcal{G},\widetilde{\mathbb{C}}),\mathcal{G}) \preceq \tau \preceq \beta_b(\mathrm{L}(\mathcal{G},\widetilde{\mathbb{C}}),\mathcal{G}) \preceq \beta(\mathrm{L}(\mathcal{G},\widetilde{\mathbb{C}}),\mathcal{G}), \tag{3.31}$$

where $\preceq$ stands for "is coarser than".

As in the theory of normed spaces we state the following result of completeness. The proof can be obtained by transferring the arguments of Proposition 3 in [45, Chapter 3] into the framework of ultra-pseudo-normed $\widetilde{\mathbb{C}}$-modules.

Proposition 3.19. *If $(\mathcal{G},\mathcal{P})$ is an ultra-pseudo-normed $\widetilde{\mathbb{C}}$-module then the dual $(L(\mathcal{G},\widetilde{\mathbb{C}}),\mathcal{P}_{L(\mathcal{G},\widetilde{\mathbb{C}})})$ is complete.*

Before proceeding with $\mathcal{G}_E$ and the topological features of its dual $\mathrm{L}(\mathcal{G}_E,\widetilde{\mathbb{C}})$ we present an easy adaptation of the Banach-Steinhaus theorem to complete ultra-pseudo-normed $\widetilde{\mathbb{C}}$-modules. As observed in Subsection 1.2 every complete ultra-pseudo-normed $\widetilde{\mathbb{C}}$-module $\mathcal{G}$ is a complete metric space with metric $d(u_1,u_2)=\mathcal{P}(u_1-u_2)$ and therefore a Baire space. This means that if $\mathcal{G}$ may be written as a countable union of closed subsets S_n then at least one S_n has nonempty interior. This fact allows us to prove the $\widetilde{\mathbb{C}}$-modules version of Osgood's theorem and the Banach-Steinhaus theorem.

Theorem 3.20. *Let $(\mathcal{G},\mathcal{P})$ be a complete ultra-pseudo-normed $\widetilde{\mathbb{C}}$-module and $(T_i)_{i\in I}$ be a family of continuous functions defined on $\mathcal{G}$ with values in $\widetilde{\mathbb{C}}$. Suppose that for each $u\in\mathcal{G}$ the family $(T_i(u))_{i\in I}$ is bounded in $\widetilde{\mathbb{C}}$. Then there exist $u_0\in\mathcal{G}$, $\eta>0$ and $C>0$ such that $|T_i(u)|_{\mathrm{e}}\le C$ for all $i\in I$ and $u\in\mathcal{G}$ with $\mathcal{P}(u-u_0)\le\eta$.*

Theorem 3.21. *Let $(\mathcal{G},\mathcal{P})$ be a complete ultra-pseudo-normed $\widetilde{\mathbb{C}}$-module and $(T_i)_{i\in I}$ be a family of functions in $L(\mathcal{G},\widetilde{\mathbb{C}})$ such that $(T_i(u))_{i\in I}$ is bounded in $\widetilde{\mathbb{C}}$ for all $u\in\mathcal{G}$. Then there exists a constant $C>0$ such that $\|T_i\|_{L(\mathcal{G},\widetilde{\mathbb{C}})}\le C$ for all $i\in I$.*

Proof. By Theorem 3.20 we may find a set $B_\eta(u_0):=\{u\in\mathcal{G}:\ \mathcal{P}(u-u_0)\le\eta\}$ and a constant $C>0$ such that $|T_iu|_{\mathrm{e}}\le C$ for all $u\in B_\eta(u_0)$ and $i\in I$. Since $[(\varepsilon^{\log(\mathcal{P}(u)/\eta)})_\varepsilon]u+u_0$ belongs to $B_\eta(u_0)$ when $u\neq 0$ in $\mathcal{G}$ it follows that

$$\begin{aligned}\frac{\eta}{\mathcal{P}(u)}|T_iu|_{\mathrm{e}} &= |T_i([(\varepsilon^{\log(\mathcal{P}(u)/\eta)})_\varepsilon]u)|_{\mathrm{e}}\\ &\le \max\{|T_i([(\varepsilon^{\log(\mathcal{P}(u)/\eta)})_\varepsilon]u+u_0)|_{\mathrm{e}},|T_iu_0|_{\mathrm{e}}\}\le C.\end{aligned}$$

Therefore $|T_iu|_{\mathrm{e}}\le (C/\eta)\mathcal{P}(u)$ for all $u\in\mathcal{G}$ and $i\in I$. This yields the uniform bound $\|T_i\|_{\mathrm{L}(\mathcal{G},\widetilde{\mathbb{C}})}\le C/\eta$. ◻

For any normed space E we already proved that $\mathrm{L}(\mathcal{G}_E,\widetilde{\mathbb{C}})$ is a complete ultra-pseudo-normed $\widetilde{\mathbb{C}}$-module for the ultra-pseudo-norm $\|\cdot\|_{\mathrm{L}(\mathcal{G}_E,\widetilde{\mathbb{C}})}$ defined by $\mathcal{P}=\|\cdot\|_{\mathcal{G}_E}$ in (3.30). The generalized functions belonging to $\mathcal{G}_{E'}$, when E' is the topological dual of E topologized through the norm $\|l\|_{E'}=\sup_{\|x\|_E=1}|l(x)|$, are particular elements of $\mathrm{L}(\mathcal{G}_E,\widetilde{\mathbb{C}})$.

Proposition 3.22. *Let E be a normed space. The map*

$$\mathcal{G}_{E'} \to \mathcal{L}(\mathcal{G}_E, \widetilde{\mathbb{C}}) : v \;\to\; (u \to v(u) := [(v_\varepsilon(u_\varepsilon))_\varepsilon]) \tag{3.32}$$

is a $\widetilde{\mathbb{C}}$-linear injection continuous with respect to $\|\cdot\|_{\mathcal{G}_{E'}}$ and $\|\cdot\|_{\mathcal{L}(\mathcal{G}_E,\widetilde{\mathbb{C}})}$.

Proof. First of all $|v_\varepsilon(u_\varepsilon)| \le \|v_\varepsilon\|_{E'}\|u_\varepsilon\|_E$ for all $(v_\varepsilon)_\varepsilon \in \mathcal{M}_{E'}$ and $(u_\varepsilon)_\varepsilon \in \mathcal{M}_E$. This implies that the map in (3.32) is well-defined, $\widetilde{\mathbb{C}}$-linear and continuous by $\|v(\cdot)\|_{\mathrm{L}(\mathcal{G}_E,\widetilde{\mathbb{C}})} = \sup_{\|u\|_{\mathcal{G}_E}=1}|v(u)|_\mathrm{e} \le \|v\|_{\mathcal{G}_{E'}}$. Concerning the injectivity, assume that $v(\cdot) = 0$ in $\mathrm{L}(\mathcal{G}_E,\widetilde{\mathbb{C}})$ but $v \neq 0$ in $\mathcal{G}_{E'}$. This means that there exists a representative $(v_\varepsilon)_\varepsilon$ of v such that $\|v_{\varepsilon_n}\|_{E'} > \varepsilon_n^q$ for some $q \in \mathbb{N}$ and a decreasing sequence $\varepsilon_n \to 0$. Therefore, we may choose a sequence $(u_n)_n \subseteq E$ with $\|u_n\|_E = 1$ for all n such that $|v_{\varepsilon_n}(u_n)| > \varepsilon_n^q$. Let now $(u_\varepsilon)_\varepsilon$ be the net in $E^{(0,1]}$ defined by 0 on $(\varepsilon_0, 1]$ and u_n on $(\varepsilon_{n+1}, \varepsilon_n]$. Clearly $(u_\varepsilon)_\varepsilon \in \mathcal{M}_E$ and by construction $(v_\varepsilon(u_\varepsilon))_\varepsilon \notin \mathcal{N}$. Therefore $v(u) \neq 0$ for $u = [(u_\varepsilon)_\varepsilon] \in \mathcal{G}_E$ which contradicts our hypothesis. $\square$

In analogy with the Hahn-Banach theorem for normed spaces we may construct an element of the dual $\mathrm{L}(\mathcal{G}_E,\widetilde{\mathbb{C}})$ having an assigned value on some $u \in \mathcal{G}_E$. In the sequel we denote the complex generalized number $[(\|u_\varepsilon\|_E)_\varepsilon]$ by $\|u\|_E$.

Proposition 3.23. *For any $u \in \mathcal{G}_E$ there exists $v \in \mathcal{G}_{E'}$ such that $\|v\|_{\mathcal{G}_{E'}} = 1$ and $v(u) = \|u\|_E$.*

Proof. Take a representative $(u_\varepsilon)_\varepsilon$ of u. By the Hahn-Banach theorem we have that for all $\varepsilon \in (0,1]$ there exists $v_\varepsilon \in E'$ such that $v_\varepsilon(u_\varepsilon) = \|u_\varepsilon\|_E$ and $\|v_\varepsilon\|_{E'} = 1$. Hence $(v_\varepsilon)_\varepsilon \in \mathcal{M}_{E'}$ and $v = [(v_\varepsilon)_\varepsilon] \in \mathcal{G}_{E'}$ satisfies the assertion. $\square$

Corollary 3.24. *For all $u \in \mathcal{G}_E$*

$$\|u\|_{\mathcal{G}_E} = \sup_{\substack{v\in\mathcal{G}_{E'},\\ \|v\|_{\mathcal{G}_{E'}}=1}} |v(u)|_\mathrm{e}. \tag{3.33}$$

Proof. The right-hand side of (3.33) is smaller than the left-hand side since the estimate $|v(u)|_\mathrm{e} \le \|v\|_{\mathcal{G}_{E'}}\|u\|_{\mathcal{G}_E}$ holds for all $u \in \mathcal{G}_E$ and $v \in \mathcal{G}_{E'}$. By Proposition 3.23 there exists $v \in \mathcal{G}_{E'}$ with ultra-pseudo-norm 1 such that $v(u) = \|u\|_E$. Then $|v(u)|_\mathrm{e} = \|u\|_{\mathcal{G}_E}$ and the equality in (3.33) is attained. $\square$

Remark 3.25. Corollary 3.24 says that $(\mathcal{G}_E, \|\cdot\|_{\mathcal{G}_E})$ is isometrically contained in $(\mathrm{L}(\mathcal{G}_{E'},\widetilde{\mathbb{C}}), \|\cdot\|_{\mathrm{L}(\mathcal{G}_{E'},\widetilde{\mathbb{C}})})$. In particular applying this result to $\mathcal{G}_F$ with $F = E'$ we have that $(\mathcal{G}_{E'}, \|\cdot\|_{\mathcal{G}_{E'}})$ is isometrically contained in $(\mathrm{L}(\mathcal{G}_E,\widetilde{\mathbb{C}}), \|\cdot\|_{\mathrm{L}(\mathcal{G}_E,\widetilde{\mathbb{C}})})$ when E is reflexive.

We conclude the paper proving the following proposition on bounded subsets.

Proposition 3.26. *Let E be a normed space. $A \subseteq \mathcal{G}_E$ is $\|\cdot\|_{\mathcal{G}_E}$-bounded if and only if it is $\sigma(\mathcal{G}_E, \mathcal{L}(\mathcal{G}_E,\widetilde{\mathbb{C}}))$-bounded.*

Proof. If A is $\|\cdot\|_{\mathcal{G}_E}$-bounded then it is $\sigma(\mathcal{G}_E, \mathrm{L}(\mathcal{G}_E, \widetilde{\mathbb{C}}))$-bounded since the topology $\sigma(\mathcal{G}_E, \mathrm{L}(\mathcal{G}_E, \widetilde{\mathbb{C}}))$ is coarser than the sharp topology on $\mathcal{G}_E$ defined by $\|\cdot\|_{\mathcal{G}_E}$. Assume now that A is $\sigma(\mathcal{G}_E, \mathrm{L}(\mathcal{G}_E, \widetilde{\mathbb{C}}))$-bounded. For all $v \in \mathcal{G}_{E'} \subseteq \mathrm{L}(\mathcal{G}_E, \widetilde{\mathbb{C}})$ we have that $\sup_{u\in A} |v(u)|_{\mathrm{e}} < +\infty$. We can interpret the set A as a family of maps in $\mathrm{L}(\mathcal{G}_{E'}, \widetilde{\mathbb{C}})$ such that for all $v \in \mathcal{G}_{E'}$ the family $(v(u))_{u\in A}$ is bounded in $\widetilde{\mathbb{C}}$. Since $\mathcal{G}_{E'}$ is complete, by Theorem 3.21 there exists $C > 0$ such that $\|u\|_{\mathrm{L}(\mathcal{G}_{E'}, \widetilde{\mathbb{C}})} \le C$ for all $u \in A$. By Corollary 3.24 we conclude that $\sup_{u\in A} \|u\|_{\mathcal{G}_E} \le C$. This means that A is $\|\cdot\|_{\mathcal{G}_E}$-bounded. $\square$

As a consequence of Proposition 3.26 one may write

$$\sigma(\mathrm{L}(\mathcal{G}_E, \widetilde{\mathbb{C}}), \mathcal{G}_E) \preceq \tau \preceq \beta_b(\mathrm{L}(\mathcal{G}_E, \widetilde{\mathbb{C}}), \mathcal{G}_E) = \beta(\mathrm{L}(\mathcal{G}_E, \widetilde{\mathbb{C}}), \mathcal{G}_E) \tag{3.34}$$

where τ is the topology of the ultra-pseudo-norm $\|\cdot\|_{\mathrm{L}(\mathcal{G}_E, \widetilde{\mathbb{C}})}$ on $\mathrm{L}(\mathcal{G}_E, \widetilde{\mathbb{C}})$.

References

[1] H. A. Biagioni. *A Nonlinear Theory of Generalized Functions.* Number 1421 in Lecture Notes in Math. Springer-Verlag, Berlin, 1990.

[2] H. Biagioni and M. Oberguggenberger. Generalized solutions to Burgers' equation. *J. Diff. Eqs.*, 97(2):263–287, 1992.

[3] H. Biagioni and M. Oberguggenberger. Generalized solutions to the Korteweg-De Vries and the regularized long-wave equations. *SIAM J. Math Anal.*, 23(4):923–940, 1992.

[4] H. A. Biagioni and J. F. Colombeau. New generalized functions and $\mathcal{C}^\infty$ functions with values in generalized complex numbers. *J. London Math. Soc. (2)*, 33(1):169–179, 1986.

[5] N. Bourbaki. *Elements of mathematics. General topology. Part 1.* Hermann, Paris, 1966.

[6] J. F. Colombeau. *New Generalized Functions and Multiplication of Distributions.* North Holland, Amsterdam, 1984.

[7] J. F. Colombeau. *Elementary Introduction to New Generalized Functions.* North-Holland Mathematics Studies 113. Elsevier Science Publishers, 1985.

[8] J. F. Colombeau. *Multiplication of Distributions. A Tool in Mathematics, Numerical Engineering and Theoretical Physics.* Number 1532 in Lecture Notes in Math. Springer-Verlag, New York, 1992.

[9] J. F. Colombeau, A. Heibig, and M. Oberguggenberger. Generalized solutions to partial differential equations of evolution type. *Acta Appl. Math.*, 45(2):115–142, 1996.

[10] J. F. Colombeau and M. Oberguggenberger. On a hyperbolic system with a compatible quadratic term: Generalized solutions, delta waves, and multiplication of distributions. *Comm. Part. Diff. Eqs.*, 15(7):905–938, 1990.

[11] N. Dapić, S. Pilipović, and D. Scarpalézos. Microlocal analysis of Colombeau's generalized functions: propagation of singularities. *Jour. d'Analyse Math.*, 75:51–66, 1998.

[12] E. Farkas, M. Grosser, M. Kunzinger, and R. Steinbauer. On the foundations of nonlinear generalized functions I. *Mem. Amer. Math. Soc.*, 153(729), 2001.

[13] C. Garetto. Pseudo-differential operators in algebras of generalized functions and global hypoellipticity. *Acta Appl. Math.*, 80(2):123–174, 2004.

[14] C. Garetto. Topological structures in Colombeau algebras II: investigation into the duals of $\mathcal{G}_{\mathrm{c}}(\Omega)$, $\mathcal{G}(\Omega)$ and $\mathcal{G}_{\mathscr{S}}(\mathbb{R}^n)$. *arXiv:math. FA/0408278*, 2004.

[15] C. Garetto, T. Gramchev, and M. Oberguggenberger. Pseudo-differential operators with generalized symbols and regularity theory. *Preprint, University of Innsbruck*, 2003.

[16] C. Garetto and G. Hörmann. Microlocal analysis of generalized functions: pseudodifferential techniques and propagation of singularities. *arXiv:math. AP/0401397*, 2004.

[17] M. Grosser. On the foundations of nonlinear generalized functions II. *Mem. Amer. Math. Soc.*, 153(729), 2001.

[18] M. Grosser, G. Hörmann, M. Kunzinger, and M. Oberguggenberger, editors. *Nonlinear Theories of Generalized Functions, Proceedings of the workshop at the Erwin Schrödinger Institute for Mathematical Physics, Vienna 1997*, Boca Raton, 1999. Chapman & Hall/CRC.

[19] M. Grosser, M. Kunzinger, M. Oberguggenberger, and R. Steinbauer. *Geometric theory of generalized functions*, volume 537 of *Mathematics and its Applications*. Kluwer, Dordrecht, 2001.

[20] G. Hörmann. Integration and microlocal analysis in Colombeau algebras. *J. Math. Anal. Appl.*, 239:332–348, 1999.

[21] G. Hörmann. First-order hyperbolic pseudodifferential equations with generalized symbols. *to appear in J. Math. Anal. Appl.*, 2004. arXiv.org e-Print math.AP/0307406.

[22] G. Hörmann. Hölder-Zygmund regularity in algebras of generalized functions. *Z. Anal. Anwendungen*, 23:139–165, 2004.

[23] G. Hörmann and M. V. de Hoop. Geophysical modeling with Colombeau functions: Microlocal properties and Zygmund regularity. *preprint*, 2001. arXiv.org e-Print math.AP/0104007.

[24] G. Hörmann and M. V. de Hoop. Microlocal analysis and global solutions of some hyperbolic equations with discontinuous coefficients. *Acta Appl. Math.*, 67:173–224, 2001.

[25] G. Hörmann and M. Kunzinger. Microlocal analysis of basic operations in Colombeau algebras. *J. Math. Anal. Appl.*, 261:254–270, 2001.

[26] G. Hörmann and M. Oberguggenberger. Elliptic regularity and solvability for partial differential equations with Colombeau coefficients. *Electron. J. Diff. Eqns.*, 2004(14):1–30, 2004.

[27] G. Hörmann, M. Oberguggenberger, and S. Pilipovic. Microlocal hypoellipticity of linear partial differential operators with generalized functions as coefficients. *arXiv:math.AP/0303248*, 2003.

[28] J. Horváth. *Topological vector spaces and distributions*. Addison-Wesley, Reading, MA, 1966.

[29] M. Kunzinger and M. Oberguggenberger. Group analysis of differential equations and generalized functions. *SIAM J. Math. Anal*, 31(6):1192–1213, 2000.

[30] F. Lafon and M. Oberguggenberger. Generalized solutions to symmetric hyperbolic systems with discontinuous coefficients: the multidimensional case. *J. Math. Anal. Appl.*, 160:93–106, 1991.

[31] A. H. Lightstone and A. Robinson. *Nonarchimedean Fields and Asymptotic Expansions*. North-Holland, Amsterdam.

[32] M. Nedeljkov, S. Pilipović, and D. Scarpalézos. *The Linear Theory of Colombeau Generalized Functions*. Pitman Research Notes in Mathematics 385. Longman Scientific & Technical, 1998.

[33] M. Oberguggenberger. Generalized solutions to semilinear hyperbolic systems. *Monatsh. Math.*, 103(2):133–144, 1987.

[34] M. Oberguggenberger. Hyperbolic systems with discontinuous coefficients: generalized solutions and a transmission problem in acoustics. *J. Math. Anal. Appl.*, 142:452–467, 1989.

[35] M. Oberguggenberger. The Carleman system with positive measures as initial data - generalized solutions. *Transport Theory Statist. Phys.*, 20(2-3):177–197, 1991.

[36] M. Oberguggenberger. *Multiplication of Distributions and Applications to Partial Differential Equations*. Pitman Research Notes in Mathematics 259. Longman Scientific & Technical, 1992.

[37] M. Oberguggenberger. Generalized functions in nonlinear models - a survey. *Nonlinear Anal.*, 47(8):5029–5040, 2001.

[38] M. Oberguggenberger and M. Kunzinger. Characterization of Colombeau generalized functions by their point values. *Math. Nachr.*, 203:147–157, 1999.

[39] M. Oberguggenberger, S. Pilipović, and D. Scarpalézos. Local properties of Colombeau generalized functions. *Math. Nachrichten*, 256:88–99, 2003.

[40] M. Oberguggenberger and T. D. Todorov. An embedding of Schwartz distributions in the algebra of asymptotic functions. *Internat. J. Math. Math. Sci.*, 21(3), 1998.

[41] M. Oberguggenberger and Y. G. Wang. Regularized conservation laws and nonlinear geometric optics. *Monatsh. Math.*, 127(1):55–66, 1999.

[42] A. P. Robertson and W. Robertson. *Topological vector spaces.* Number 53 in Cambridge Tracts in Mathematics. Cambridge University Press, Cambridge-New York, 1980.

[43] A. Robinson. *Nonstandard Analysis.* North Holland, Amsterdam, 1966.

[44] A. Robinson. Function theory on some nonarchimedean fields. *Amer. Math. Montly*, 80(6):87–109, 1973. Part II: Papers in the Foundations of Mathematics.

[45] H. L. Royden. *Real Analysis.* Macmillan Publishing Company, New York, 1988.

[46] D. Scarpalézos. Topologies dans les espaces de nouvelles fonctions généralisées de Colombeau. $\widetilde{\mathbb{C}}$-modules topologiques. Université Paris 7, 1992.

[47] D. Scarpalézos. Some remarks on functoriality of Colombeau's construction; topological and microlocal aspects and applications. *Integral Transform. Spec. Funct.*, 6(1-4):295–307, 1998.

[48] D. Scarpalézos. Colombeau's generalized functions: topological structures; microlocal properties. a simplified point of view. *I. Bull. Cl. Sci. Math. Nat. Sci. Math.*, (25):89–114, 2000.

[49] H. H. Schaefer and M. P. Wolff. *Topological vector spaces.* Number 3 in Graduate Texts in Mathematics. Springer-Verlag, New York, 1999.

[50] T. Todorov and R. S. Wolf. Hahn field representation of Robinson's asymptotic numbers. *to appear in the proceedings of the conference on generalized functions, Guadalupe*, 2000.